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MATHEMATICS MAGAZINE

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(Continued on the inside of the back cover.)

ON THE STABILITY OF DIFFERENTIAL EXPRESSIONS

S. M. Ulam and D. H. Hyers

1. *Introduction*

Every student of calculus knows that two functions may differ uniformly by a small amount and yet their derivatives may differ widely. On the other hand, it is more or less obvious intuitively, and quite easily proved, that if a continuous function f on a finite closed interval has a proper maximum at a point $x = a$ then any continuous function g sufficiently close to f also has a maximum arbitrarily close to $x = a$. When f and g have first derivatives, this means that f' vanishes and changes sign at $x = a$ while g' vanishes at some point of a preassigned neighborhood of $x = a$ (see theorem 1 for the case $n = 1$).

Thus, in spite of the first sentence of the above paragraph, there are certain cases in which the derivatives cannot "differ much" *providing* however that we are allowed to *shift* the point a little at which the derivative is taken. This paper gives a few elementary results along these lines. Later we plan to develop these ideas further and give various applications.

2. *Some Theorems for One Independent Variable*

In this section we shall use the term "neighborhood of a point" to mean an interval whose center is the point. Its "radius" is half the length of the interval.

Theorem 1: *Let $f(x)$ be a function having an nth derivative in a neighborhood N of the point $x = a$. If $f^{(n)}(a) = 0$, and $f^{(n)}(x)$ changes sign at $x = a$, then corresponding to each $\varepsilon > 0$ there exista a $\delta > 0$ such that, for each function $g(x)$ having an nth derivative in N and satisfying the inequality $|g(x) - f(x)| < \delta$ in N, there exists a point $x = b$ such that $g^{(n)}(b) = 0$ and $|b - a| < \varepsilon$.*

Proof: Let ϵ be chosen (without loss of generality) less than the radius of N, and assume that $f^{(n)}(x)$ is negative immediately to the left of $x = a$ and positive to the right (the opposite case is handled similarly). Then there exist points x_1 and x_2 such that $f^{(n)}(x_1) < 0$, $f^{(n)}(x_2) > 0$, $|x_1 - a| < \varepsilon/2$, $|x_2 - a| < \varepsilon/2$. Let $h > 0$ be chosen so that $h < \varepsilon/2n$ and also so that $\Delta_h^n f(x_1) < 0$ and $\Delta_h^n f(x_2) > 0$, and keep h fixed. Let $\delta > 0$ be chosen to be smaller than $(|\Delta_h^n f(x_1)|)/2^n$ and $(|\Delta_h^n f(x_2)|)/2^n$. Now, if $g(x)$ is any nth-differentiable function defined for $x \in N$ and such that $|g(x) - f(x)| < \delta$ for $x \in N$, we form the nth difference of g at x_1 and at x_2 and obtain:

$$\begin{aligned}|\Delta_h^n g(x_i) - \Delta_h^n f(x_i)| \le\ & |g(x_i + nh) - f(x_i + nh)| \\ & + n|g(x_i + (n-1)h) - f(x_i + (n-1)h)| \\ & + \binom{n}{2}|g(x_i + (n-2)h) - f(x_i + (n-2)h)| + \cdots \\ & + |g(x_i) - f(x_i)|, \qquad i = 1,2.\end{aligned}$$

By the choice of h and ε it follows that $x_i + rh \in N$, $r = 0,1,2,\cdots,n$, so that

$$|\Delta_h^n g(x_i) - \Delta_h^n f(x_i)| < \delta[1 + \tbinom{n}{1} + \tbinom{n}{2} + \cdots + \tbinom{n}{n-1} + 1] = 2^n\delta.$$

By the choice of δ it follows that

$$\Delta_h^n g(x_1) < 0 \quad \text{and} \quad \Delta_h^n g(x_2) > 0$$

Now by the law of the mean we have

$$\Delta_h^n g(x_1) = h^n g^{(n)}(x_1 + n\theta_1 h), \quad \text{where} \quad 0 < \theta_1 < 1$$

$$\Delta_h^n g(x_2) = h^n g^{(n)}(x_2 + n\theta_2 h), \quad \text{where} \quad 0 < \theta_2 < 1.$$

Since $g^{(n)}(x)$ is a derived function* it follows that $g^{(n)}(b) = 0$ for some b between $x_1 + n\theta_1 h$ and $x_2 + n\theta_2 h$. Now $x_1 < b < x_2 + nh < x_2 + \varepsilon/2$, and $|x_i - a| < \varepsilon/2$, $i = 1,2$, so that $|b - a| < \varepsilon$.

Theorem 2: *Let $F(x,y)$ be a continuous function of two variables in a region R containing the point (a,b), and let $f(x)$ be a function with a continuous derivative $f'(x)$ in the neighborhood N of the point a and such that $f'(x) - F(x,f(x))$ vanishes at a and changes sign in every neighborhood of the point a, where $b = f(a)$. Then corresponding to each $\varepsilon > 0$ there is a $\delta > 0$ such that, for each differentiable function $g(x)$ satisfying the inequality $|f(x) - g(x)| < \delta$, $(x \in N)$, there exists a point x_0 such that $g'(x_0) = F(x_0, g(x_0))$, and such that $|x_0 - a| < \varepsilon$.*

Proof: Choose x_1 and x_2, within N and also within the neighborhood $|x - a| < \varepsilon$, such that

$$\phi(x_1') = \alpha_1 = f'(x_1') - F(x_1', f(x_1')) < 0$$

$$\phi(x_2') = \alpha_2 = f'(x_2') - F(x_2'\ f(x_2')) > 0$$

Since the (proper, relative) maximum and minimum values of the continuous function $\phi(x)$ form a finite or enumerably infinite set**, it is possible

* See for example Franklin, A Treatise on Advanced Calculus, p. 117.
** Cf. Hobson, Theory of Functions of a Real Variable, vol. I, p. 329.

to select x_1', x_2' such that $\phi(x)$ does not have a maximum or minimum value at x_1' or at x_2'. Hence in every neighborhood of x_1' there are points ξ and η such that $\phi(\xi) > \phi(x_1') > \phi(\eta)$ and similarly for x_2'. Hence, if we put $C_1 = \alpha_1 + F(x_1', f(x_1'))$, the function $f'(x) - C_1$ vanishes at $x = x_1'$ and changes sign in every neighborhood of $x = x_1'$. Let η_1 be chosen so that the interval $|x - x_1'| < \eta_1$ is within N and also within the neighborhood $|x - a| < \varepsilon$, and also so that $|F(x', y') - F(x, y)| < |\alpha|/2$ for $|x' - x| < \eta_1$ and $|y' - y| < \eta_1$. Since $f(x)$ is continuous, there exists $\varepsilon_1 > 0$ such that $\varepsilon_1 < \eta_1$ and such that $|f(x') - f(x)| < \eta/2$ for $|x' - x| < \varepsilon_1$ (x and $x' \in N$). By theorem 1, there is a $\delta_1 > 0$, which may be chosen so that $\delta_1 < \eta_1/2$, so that for any differentiable function $g(x)$ defined on N, with $|g(x) - f(x)| < \delta_1$ for $x \in N$, there exists a point $x = x_1$ such that $|x_1' - x_1| < \varepsilon_1$ and such that $g'(x_1) = C_1 = \alpha_1 + F(x_1', f(x_1'))$. Hence

$$|g'(x_1) - F(x_1, g(x_1)) - \alpha_1| = |F(x_1', f(x_1')) - F(x_1, g(x_1))| < |\alpha|/2,$$

since $|x_1' - x_1| < \varepsilon_1 < \eta_1$ and $|f(x_1') - g(x_1)| \leq |f(x_1') - f(x_1)| + |f(x_1) - g(x_1)| < \eta_1/2 + \delta_1 < \eta_1$. Therefore $g'(x_1) - F(x_1, g(x_1)) < \alpha_1/2 < 0$, where x_1 is a point in N such that $|x_1 - a| < \varepsilon$. Similarly, it can be shown that there exists a point x_2 in N such that $|x_2 - a| < \varepsilon$ and $g'(x_2) - F(x_2, g(x_2)) > \alpha_2/2 > 0$. Since $g'(x) - F(x, g(x))$ is a derived function it takes on the value 0 at some point x_0 between x_1 and x_2. Hence $g'(x_0) = F(x_0, g(x_0))$, where $|g(x) - f(x)| < \delta$ and $|x_0 - a| < \varepsilon$.

3. *A Two Dimensional Theorem.*

Theorem 3. *Let $F(x,y)$ be continuous with continuous second partial derivatives in a neighborhood* N of a point (x_0, y_0), and let us suppose that*

(i) $$F_x(x_0, y_0) = F_y(x_0, y_0) = 0$$

(ii) $$F_{yy}(x_0, y_0) \neq 0 \quad \text{and} \quad F_{xy}^2 - F_{xx}F_{yy} \neq 0$$

at (x_0, y_0). Then, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that, if $G(x,y)$ is any function with continuous second partial derivatives and with $G_{yy} \neq 0$ on N, satisfying the inequality

(iii) $$|G(x,y) - F(x,y)| < \delta$$

on N, then there is a point (x_1, y_1) such that $G_x(x_1, y_1) = G_y(x_1, y_1) = 0$, where $|x_1 - x_0| < \varepsilon$, $|y_1 - y_0| < \varepsilon$.

Proof: Consider the equation

*Here N denotes a two dimensional neighborhood, e.g. a square centered at the point in question.

$$(1) \qquad F_y(x,y) = 0.$$

Since it is satisfied for $x=x_0$, $y=y_0$ and since $F_{yy}\neq 0$ at (x_0,y_0), there exists a solution $y=\varphi(x)$ of Eq. (1) passing through the point (x_0,y_0) and having a continuous derivative, for x in some neighborhood $|x-x_0|\leq a$ of the point x_0.

Given $\varepsilon>0$, choose $\varepsilon_1>0$ so that $|\varphi(x)-\varphi(x_0)|<\varepsilon/2$ for $|x-x_0|<\varepsilon_1$, where $\varepsilon_1<\varepsilon$.

Now consider the function

$$\Phi(x) = F(x,\varphi(x)),$$

which has continuous first and second derivatives

$$\Phi'(x) = F_x(x,\varphi(x))$$

$$\Phi''(x) = \left.\frac{F_{xx}F_{yy} - F_{xy}^{\;2}}{F_{yy}}\right|_{y=\varphi(x)}$$

By hypothesis, $\Phi''(x_0)\neq 0$. Hence $\Phi'(x)$ vanishes and changes sign at $x=x_0$, so by theorem 1, there exists $\delta_1>0$ such that, if $g(x)$ is any differentiable function with $|g(x)-\Phi(x)|<\delta_1$ for $|x-x_0|\leq a$, then $g'(x_1)=0$ for some point x_1 satisfying $|x_1-x_0|<\varepsilon_1$.

By hypothesis $F(x,y)$ is continuous so it is continuous in y, uniformly with respect to x for $|x-x_0|\leq a$. Let $\varepsilon_2<\varepsilon/2$ be chosen so that

$$|F(x,y_1) - F(x,y_2)| < \delta_1/2$$

for $|y_1-y_2|<\varepsilon_2$ and all x satisfying $|x-x_0|\leq a$, where it is understood that $(x,y_i)\in N$, $i=1,2$.

By theorem 1 it follows that for each fixed x in the interval I: $|x-x_0|\leq a$ there is a $\delta(x)$ such that for any differentiable function $h(y)$ satisfying $|h(y)-F(x,y)|<\delta(x)$, $y\in N$, there exists y_1 such that $|y_1-\varphi(x)|<\varepsilon_2$ and $h'(y_1)=0$. The closed interval I may be covered with intervals of length $\delta(x)$ and by the Heine-Borel theorem there exists a finite subcovering and hence a δ_I independent of x. Now let $G(x,y)$ be any function defined on N with continuous second derivatives and with $G_{yy}\neq 0$ on N, such that inequality (iii) is satisfied, where δ is the smaller of δ_I and $\delta_1/2$. Then there exists a function $y=\varphi_1(x)$ such that $|\varphi(x)-\varphi_1(x)|<\varepsilon_2$ and $G_y(x,\varphi_1(x))=0$ for $|x-x_0|\leq a$. Since $G_{yy}\neq 0$, $y=\varphi_1(x)$ is differentiable. Hence the function $\Psi(x)=G(x,\varphi_1(x))$ is differentiable and $\Psi'(x)=G_x(x,\varphi_1(x))$.

Now

$$|\Psi(x)-\Phi(x)| \leq |G(x,\varphi_1(x))-F(x,\varphi_1(x))| + |F(x,\varphi_1(x))-F(x,\varphi(x))|$$

The first term on the right is less than $\delta \le \delta_1/2$, and since $|\varphi_1(x)-\varphi(x)| < \varepsilon_2$, for $|x-x_0| \le a$ we have $|F(x,\varphi_1(x)-F(x,\varphi(x))| < \delta_1/2$. Therefore $|\Psi(x)-\Phi(x)| < \delta_1$ for $|x-x_0| \le a$. Hence $\Psi(x)$ is a $g(x)$, and there exists a point x_1 with $|x_1-x_0| < \varepsilon_1$ such that

$$\Psi'(x_1) = G_x(x_1,\varphi_1(x_1)) = 0.$$

Putting $y_1 = \varphi_1(x_1)$ we have $G_x(x_1,y_1)=0$, $G_y(x_1,y_1)=0$ and $|x_1-x_0| < \varepsilon_1 < \varepsilon$, while

$$|y_1-y_0| \le |\varphi_1(x_1)-\varphi(x_1)| + |\varphi(x_1)-\varphi(x_0)| < \varepsilon_2+\varepsilon/2 < \varepsilon,$$

so the theorem is proved.

4. *Some Counter-Examples.*

1. The following example is due to John von Neumann.

Take $F(x,y,y',y'') = y'^2 - yy'' - x = 0$, $f(x) \equiv 0$, $x_0 = 0$. Then $F(x,f(x), f'(x),f''(x)) = -x$ obviously changes sign as x passes through the value x_0. Now take $g(x) = \delta \sin(x/\delta)$ where δ is any positive number, and take $\varepsilon = \frac{1}{2}$. Obviously $|g(x)-f(x)| < \delta$ for all x. Since $g'(x) = \cos(x/\delta)$, $g''(x) = -(1/\delta)\sin(x/\delta)$, $F(x,g(x),g'(x),g''(x)) = \cos^2(x/\delta)+\sin^2(x/\delta) - x = 1-x$. Thus the root of $F(x,g(x),g'(x),g''(x)) = 0$ is at $x=1$, which is at a distance $1 > \frac{1}{2}$ from the root $x=0$ of $F(x,f(x),f'(x),f''(x))$. The stability theorem fails in this case. One must be careful about formulating "implicit" stability theorems in derivatives.

2. In the two dimensional case, one has to be careful about formulating the condition of "change of sign", as the following example shows. Geometrically, it amounts essentially to taking a torus with axis OZ, and tipping it slightly.

Consider the function

$$z = F(x,y) = \sqrt{b^2-a^2+2a\sqrt{x^2+y^2}-x^2-y^2}, \quad (a > b) \quad .$$

in the neighborhood of the point $(a,0)$. On differentiating we have:

$$F_x = \frac{x}{z}\left[\frac{a}{\sqrt{x^2+y^2}} - 1\right]$$

$$F_y = \frac{y}{z}\left[\frac{a}{\sqrt{x^2+y^2}} - 1\right]$$

Clearly F_x changes sign from + to - as x increases through $x=a$ and y

is held constant $(y=0)$. Also F_y changes sign from + to - as y increases through $y=0$ and x is held fixed $(x=a)$.

Now consider the following function

$$z = G(x,y) = \delta y + F(x,y),$$

where δ is a small positive number. Then for $|x-a| < b/2$ and $|y| < b/2$, $|G(x,y) - F(x,y)| < \delta$. We have

$$G_x = \frac{x}{F(x,y)}\left[\frac{a}{\sqrt{x^2+y^2}} - 1\right] = F_x$$

$$G_y = \delta + \frac{y}{F(x,y)}\left[\frac{a}{\sqrt{x^2+y^2}} - 1\right].$$

The locus of zeros of G_x in the neighborhood of $(a,0)$ is obviously the circle $x^2+y^2=a^2$. But along this circle, $G_y = \delta > 0$. Hence there is no point near $(a,0)$ where both $G_x = 0$ and $G_y = 0$.

Los Alamos Scientific Laboratory and

University of Southern California

The Concept - Infinity - Aborning

How long to count to one million?

Make a quick guess, then read this note.

A little boy asked me "What is the end of numbers?". The answer was, of course, "There is no end, for you can always add 1 to any number you have and get the next one, just as you can add 1 to 3 and get 4". But this non-terminating process worried him, as it does everybody. So he followed with,"How long would it take to count to one million?" A little timing and some simple arithmetic showed that it would take over forty days of eight hours each (with no short cuts). Then he asked "How long to count to a billion?" When we had found that it would take over one hundred and thirty three years, he got a pencil and paper saying, "I am going to take some notes on this." And began, "My Uncle, one of the World's greatest mathematicians (hic!) says ··· " Since then I have asked several persons, some schooled in mathematics and some not, the latter question. Most of the answers were such as "several days" or "several years." Only one was correct, that was, "I don't know, but you can never do it".

Editor

AREAS SWEPT OUT BY TANGENT LINE SEGMENTS

Preston C. Hammer

INTRODUCTION.

The author happened upon the formulas for area presented in this paper when working on an ergodic problem proposed to him by Professor B. H. Neumann. A search of the literature reveals that the planar case implies Guldin's Formula and is practically equivalent to it. However, as far as we have been able to determine, the particular viewpoint we have chosen is either unknown or is little known to mathematicians in this country.

Briefly put, we shall develop a formula which makes possible the composition of curves in the polar coordinate plane and we shall give an area-preserving mapping of a developable surface onto the polar plane which is not the customary isometric mapping.

There seems no good reason that the planar case should not be presented in elementary calculus since a proof can be given on that level. We shall, however, give a proof which holds for two or three (actually n) dimensions.

THE AREA FORMULA.

Let $\vec{x} = \vec{x}(s)$ be the vector equation of a space curve, C, with the parameter $s \geq 0$ taken as the arc length from some point on the curve. Let C be such that the derivative vectors $\vec{x}'(s)$ and $\vec{x}''(s)$ do not vanish at any point and such that $\vec{x}''(s)$ is continuous. Now, the accumulated planar angle, α, between tangent vectors to C may be written

$$\alpha = \int_0^s |\vec{x}''(s)|\, ds,$$

where $|\vec{x}|$ is the length of $\vec{x}$. In differential form,

$$d\alpha = |\vec{x}''(s)|\, ds.$$

Note that $|\vec{x}''(s)|$ is the principal curvature of C. Now from our assumptions $|\vec{x}''(s)| > 0$ whence α is a continuous increasing function of s and therefore α may be used as a parameter for C. Now, let $\rho(s) \geq 0$ be a continuous function of s (and hence of α) and lay off along the direction of $\vec{x}'(s)$ from the point of tangency on C the line segment of length $\rho(s)$. Then we state

THEOREM 1. *The cumulative positive area, A, which the tangent line segments to C of length $\rho(s)$ sweep out is given by*

$$A = \tfrac{1}{2}\int_0^{\alpha}[\rho(\alpha)]^2\, d\alpha.$$

Before proceeding with the proof, we observe that if the line segments described cover a portion of the developable surface tangential to C twice, say, that area is "counted" twice in A. That is the reason that we say *cumulative* positive area. The salient feature of the formula for A is that it is *independent of the* curve C.

Proof. Let $\vec{z}$ describe a surface in space with parameters u and v. Then it is well known that with adequate restrictions on $\vec{z}(u,v)$ the area of the surface is given by a double integral over the appropriate region of the (u,v)-plane, namely:

$$\iint \sqrt{|\vec{z}_u|^2 \cdot |z_v|^2 - (\vec{z}_u,\vec{z}_v)^2}\, du\, dv$$

where subscripts refer to partial differentiation and $(\vec{z}_u,\vec{z}_v)$ is the scalar product of the two vectors.

The surface in which we are interested may be written

$$\vec{z}(\rho,s) = \vec{x}(s) + \rho\vec{x}'(s)$$

where ρ varies from 0 to $\rho(s)$ and s varies from 0 to an arbitrary positive value. Substituting in the above area formula with $\rho = u$ and $s = v$ we have

$$A = \iint \sqrt{|\vec{x}'(s)|^2\, |\vec{x}'(s)+\rho\vec{x}''(s)|^2 - (\vec{x}'(s),\vec{x}'(s)+\rho\vec{x}''(s))^2}\, d\rho ds.$$

Now $|\vec{x}'(s)| = 1$ always and $(\vec{x}'(s),\vec{x}''(s)) = 0$ since $\vec{x}'(s)$ and $\vec{x}''(s)$ are perpendicular vectors. Using these facts we find

$$A = \iint \rho\,|\vec{x}''(s)|\, d\rho\, ds.$$

Now

$$\int_0^{\rho(s)} \rho\, d\rho = \tfrac{1}{2}\rho^2(s) \quad \text{and} \quad |\vec{x}''(s)|\, ds = d\alpha.$$

Hence we have with $\rho(s) = \rho(\alpha)$,

$$A = \tfrac{1}{2}\int_0^{\alpha} \rho^2(\alpha)\, d\alpha.$$

Q.E.D.

Now consider the transformation T from the developable surface

$$\vec{z} = \vec{x}(s) + \rho\vec{x}\,'(s)$$

to the polar (ρ,α)-plane where the image of $\vec{z}$ is (ρ,α) with

$$\alpha = \int_0^s |\vec{x}\,''(s)|\, ds.$$

Then we have

THEOREM 2. *The transformatjon T is an area-preserving transformation from the surface $\vec{z} = \vec{z}(\rho, s)$ to the polar plane. Under T the curve C is mapped into the pole.*

This theorem is an immediate consequence of Theorem 1 since the expression for the area A is the same as that in polar coordinates. If α should increase beyond 2π, the cumulative area in the polar plane is naturally used.

THE PLANAR CASE.

In case the curve C of the preceding section should be planar then α is the accumulated positive angular deviation of the tangent vectors from a fixed direction vector. In case C is a closed convex curve with a unique tangent vector at each point then α may range from 0 to 2π. Then the area swept out by tangent line segments of length $\rho(\alpha) \geq 0$ is

$$\tfrac{1}{2} \int_0^{2\pi} \rho^2(\alpha)\, d\alpha.$$

Observe that this formula is *independent* of C as long as C is convex; it depends only on the length function $\rho(\alpha)$. In particular, if C is shrunk to a point, we have the usual polar coordinate area formula. If we can evaluate the integral of $\rho^2(\alpha)$ we have determined the area between a closed convex curve C and its polar *composition* with $\rho = \rho(\alpha)$ formed by tangent line segments to C of length $\rho(\alpha)$ along tangent vectors making an angle α with the half-line $\rho > 0$, $\alpha = 0$.

The formula applies to arbitrary closed convex planar curves using vectors of contact for each α pointing to one sense. Again in the planar case, one may, if C has a flex point, let α be algebraically summed. The effect of this on Theorem 1 would be to change "cumulative positive area" to "net accumulated area"—and the area could then be negative.

We shall now give examples of the use of these considerations to establishing values of definite integrals. Let $y = ln(x)$ represent the logarithmic curve in the (x,y)-plane and let the tangent vector be one with positive x-component. On this vector lay off a constant positive length a from the point of tangency. Letting (u,v) be the endpoints of this line segment, we have

$$u = x\left[1 + \frac{a}{\sqrt{1+x^2}}\right]$$

$$v = \ln(x) + \left[\frac{a}{\sqrt{1+x^2}}\right]$$

Now, the angle of the tangent vector changes through $\pi/2$ as x goes from 0 to ∞. Hence the area swept out by the tangent line segments of length a is $(\pi a^2)/4$. On the other hand, the same area is given by

$$\int_0^\infty (v - \ln(u))\, du = \frac{\pi a^2}{4}$$

Substituting for v, u, du in terms of x, we have

$$\int_0^\infty \left[\frac{a}{\sqrt{1+x^2}} - \ln\left[1 + \frac{a}{\sqrt{1+x^2}}\right]\right] \left[1 + \frac{a}{(1+x^2)^{3/2}}\right] dx = \frac{\pi a^2}{4}$$

Now, we let $x = \tan\theta$ and we have

$$\int_0^{\pi/2} [a\cos\theta - \ln(1 + a\cos\theta)]\,[1 + a\cos^3\theta]\sec^2\theta\, d\theta = \frac{\pi a^2}{4}$$

Again letting $z = \cos\theta$, we find

$$\int_0^1 \frac{[az - \ln(1+az)]}{z^2} \frac{[1 + az^3]}{\sqrt{1-z^2}}\, dz = \frac{\pi a^2}{4}$$

More generally, if $y = f(x)$ is a function defined for $-\infty < x < \infty$ and if $f''(x) < 0$ and if we lay off tangent line segments of length $\rho(\alpha) = \rho(x)$ from the point of tangency, then these line segments sweep out an area $\frac{1}{2}\int_{\alpha_0}^{\alpha_1} \rho^2(\alpha)\,d\alpha$ where α_0 is the limiting tangent angle as $x \to -\infty$ and α_1 is the limiting tangent angle as $x \to +\infty$. Then the endpoint (u,v) of the tangent line segment is given by

$$u = x + \frac{\rho}{\sqrt{1+y'^2}}, \qquad v = f(x) + \frac{\rho y'}{\sqrt{1+y'^2}}$$

and

$$\int_{-\infty}^{+\infty} [v - f(u)]\, du = \frac{1}{2}\int_{\alpha_0}^{\alpha_1} \rho^2(\alpha)\, d\alpha.$$

One other simple type of application leads to values of definite integrals in some cases. Let $y = f(x)$ represent a curve such that $f(x) > 0$, $f'(x) < 0$, $f''(x) > 0$ for $x \geq a$. Then, the area swept out by the tangent

lines to the curve bounded by the curve, the x-axis and the tangent line at $x=a$ is given by

$$\tfrac{1}{2}\int_a^{\infty}(y/y')^2 y''\,dx\,.$$

This integral exists if $\int_a^{\infty} y\,dx$ does and the other conditions hold. (See Figure 1.)

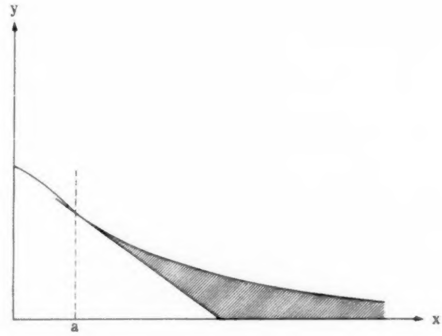


Figure 1.

This formula holds since the tangent length to the x-axis is

$$\rho = (y/y')\sqrt{1+y'^2},$$

and $\alpha = \text{arc tan } y'$, hence

$$d\alpha = \frac{y''}{1+y'^2}\,dx.$$

Therefore

$$\tfrac{1}{2}\int \rho^2\,d\alpha = \tfrac{1}{2}\int_a^{\infty}(y/y')^2 y''\,dx\,.$$

For example, if $y = \dfrac{1}{\sqrt{2\pi}}e^{-x^2/2}$, one may use a knowledge of the integral of this "normal" curve to find an exprrssion for

$$\int_1^{\infty}\frac{e^{-x^2/2}}{x^2}\,dx$$

by taking $a=1$ in the above formula. This integral, may, of course, be evaluated by other means.

We have not deeply explored the possibilities of integral evaluation. The examples, however, show that certain definite integrals may be evaluated and suggests that familiarity with the method may lead to interesting applications.

GULDIN'S FORMULA.

Let a family of line segments simply cover a region R in the plane. Let the parameter of the family be considered as the time t, let the length of the line segment be $f(t)$, and the normal component of the velocity of the midpoint be $v(t)$ in magnitude. Then if t varies from t_0 to t_1 to cover R, Guldin's formula states that the area A of R is

$$A = \int_{t_0}^{t_1} f(t)\, v(t)\, dt.$$

Suppose that the extended line segments form a line family which has a curve C as an envelope where C has no flex point for $t_0 \leq t \leq t_1$. Then, in our terminology,

$$A = \tfrac{1}{2} \int_{\alpha_0}^{\alpha_1} [\rho_2^2(\alpha) - \rho_1^2(\alpha)]\, d\alpha$$

where $\rho_2(\alpha) - \rho_1(\alpha) = f(t)$. Then

$$\tfrac{1}{2}\,[\rho_2(\alpha) + \rho_1(\alpha)]\, d\alpha \quad \text{is} \quad v(t)\, dt.$$

Thus, in case the family of extended line segments has a envelope as described, our formula is equivalent to Guldin's Formula. Guldin's Formula also applies to the case in which the line segments are parallel.

CONCLUDING REMARKS.

The development we have given here for the planar areas can be stated in terms of the outwardly simple line families we have introduced elsewhere. However, simplicity is then lost since one must deal with Lebesque integrals and more advanced concepts of area. We will present these more technical proofs elsewhere.

The questions of negative values of ρ in space and in the plane, and of degenerate cases may be answered without difficulty. For that reason we have omitted such questions.

University of Wisconsin

TEACHING OF MATHEMATICS

Edited by

Joseph Seidlin and C. N. Shuster

This department is devoted to the teaching of mathematics. Thus articles on methodology, exposition, curriculum, tests and measurements, and any other topic related to teaching, are invited. Papers on any subject in which you, *as a teacher*, are interested, or questions which you would like others to discuss, should be sent to Joseph Seidlin, Alfred University, Alfred, New York.

EUCLID'S ALGORITHM AND ITS APPLICATIONS

Harry S. Clair

INTRODUCTION

In Book 7 of his Elements, Euclid describes a process for finding the greatest common divisor (g.c.d.) of two rational integers which is known today as "Euclid's Algorithm". Although this process may seem trivial, it is nevertheless of fundamental importance in Number Theory and Algebra, and is susceptible to great generalization. The purpose of this paper is to apply the algorithm to several elementary topics in Number Theory and Algebra. Although the topics discussed are well known in the literature, it seemed to the writer worth while to discuss them under a central theme – the Algorithm of Euclid.

1.

THE G.C.D. OF TWO RATIONAL INTEGERS

As an example, let us consider the problem of finding the g.c.d. of 301 and 133. By Euclid's algorithm, we divide the larger by the smaller to obtain a remainder less than 133:

$$301 - 2 \times 133 = 35$$

Then any divisor of 301 and 133 must also divide 35. The g.c.d. of 301 and 133 is therefore also the g.c.d. of 133 and 35. Proceeding in this manner:

$$133 - 3 \times 35 = 28$$

$$35 - 1 \times 28 = 7$$

$$28 - 4 \times 7 = 0.$$

Thus 7 is the required g.c.d.

More generally, let $a \geq b \geq 0$. Then the algorithm yields the equations:

$$a = q_1 b + r_1 \qquad (0 \leq r_1 < b)$$

$$b = q_2 r_1 + r_2 \qquad (0 \leq r_2 < r_1)$$

$$r_1 = q_3 r_2 + r_3 \qquad (0 \leq r_3 < r_2)$$

and so on. Since the non-negative integers $b, r_1, r_2, \cdots$ form a decreasing sequence of rational integers, the process ultimately leads to a remainder which is zero. Let r_{n+1} be the first zero remainder. Then the last two steps of the process will be:

$$r_{n-2} = q_n r_{n-1} + r_n \qquad (0 < r_n < r_{n-1})$$

$$r_{n-1} = q_{n+1} r_n,$$

and r_n is the required g.c.d. For by the last equation, r_n is a divisor of r_{n-1}, then by the next to the last equation r_n is a divisor of r_{n-2}. Proceeding in this manner backwards, we finally obtain the required result. It is to be noticed that this process not only proves the existence of a g.c.d. of two integers, but also shows how it can be calculated. This result can be stated thus:

Theorem 1. *If a and b are rational integers there exists an integer $d \equiv (a,b)$ called the g.c.d. of a and b such that d divides a and b, and integers p and q can be found such that*

1.1 $$pa + qb = d.$$

The special case when a and b are prime to each other, i.e., $d = 1$, is especially important. We state it as:

Corollary 1. *If a and b are prime to each other, integers p and q can be found such that*

1.2 $$pa + qb = 1.$$

It is clear from equation 1.1 that any common factor d_0 of a and b must be a factor of d.

2.

LINEAR DIOPHANTINE EQUATIONS

An equation of the form

2.1 $$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = k$$

with integral coefficients $a_1, a_2, \cdots, a_n, k$ is called "linear Diophan-

tine". Obviously no solution of this equation exists in integers unless the g.c.d. of $a_1, a_2, \cdots, a_n$ is a divisor of k. Euclid's algorithm can be used in the solution of such equations as will be illustrated in the following

Example: Find all solutions in integers of

$$37x + 23y = 752. \tag{2.2}$$

Euclid's algorithm yields the identity $5 \times 37 - 8 \times 23 = 1$. Hence $x_0 = 5 \times 752$, $y_0 = -8 \times 752$ is clearly one solution of 2.2. Now suppose x, y is an arbitrary integral solution of 2.2. Then $37x + 23y = 752 = 37x_0 + 23y_0$, and $37(x - x_0) = -23(y - y_0)$. It follows that -23 is a divisor of $(x - x_0)$. Therefore $x - x_0 = -23t$ where t is an arbitrary integer, and $y - y_0 = 37t$ We thus have as a general solution of 2.2,

$$x = 5 \times 752 - 23t, \quad y = -8 \times 752 + 37t. \tag{2.3}$$

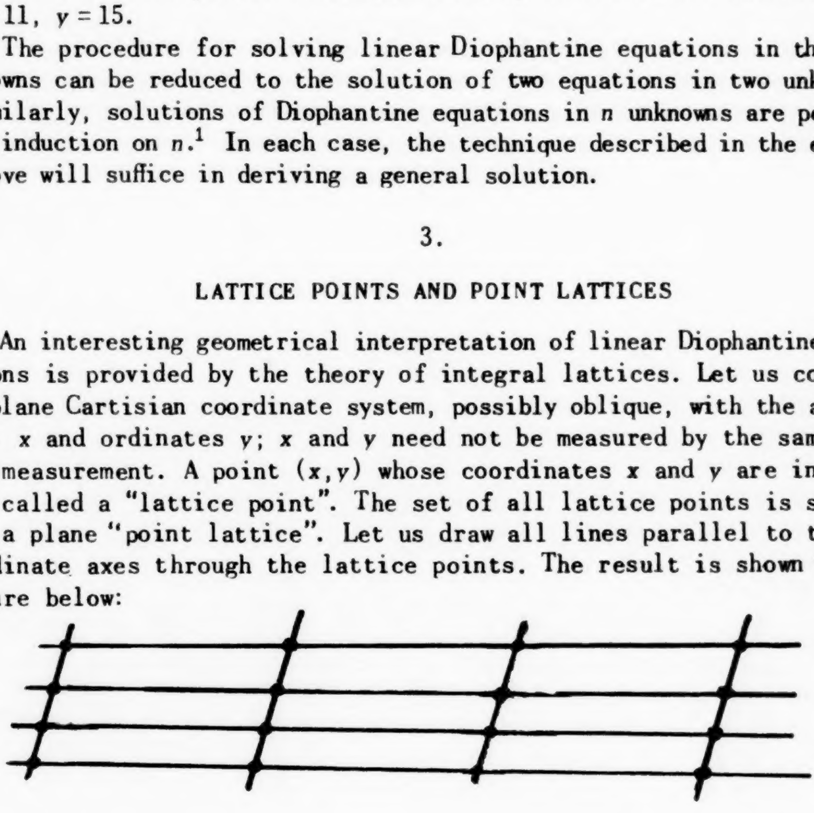
To find the least positive solution of 2.2, we set $t = 163$ and arrive at $x = 11$, $y = 15$.

The procedure for solving linear Diophantine equations in three unknowns can be reduced to the solution of two equations in two unknowns. Similarly, solutions of Diophantine equations in n unknowns are possible by induction on n.[1] In each case, the technique described in the example above will suffice in deriving a general solution.

3.

LATTICE POINTS AND POINT LATTICES

An interesting geometrical interpretation of linear Diophantine equations is provided by the theory of integral lattices. Let us consider a plane Cartisian coordinate system, possibly oblique, with the abscissas x and ordinates y; x and y need not be measured by the same unit of measurement. A point (x, y) whose coordinates x and y are integers is called a "lattice point". The set of all lattice points is said to be a plane "point lattice". Let us draw all lines parallel to the coordinate axes through the lattice points. The result is shown in the figure below:



1 For details of this method, see Trygve Nagell's "Introduction to Number Theory", John Wiley & Sons, Inc., New York, 1951, pp. 29-32.

The equation $ax+by=k$ represents a straight line. The problem of solving this equation in integers x and y is equivalent to the problem of finding the lattice points situated on the straight line in question.

Suppose that the straight line has integral coefficients, or what is obviously the same, rational coefficients. Then, by Section 2, the straight line passes through an infinity of lattice points if it passes through one lattice point. This conclusion is not valid for equations with irrational coefficients a,b,k except when there exists a positive number p such that the numbers ap, bp, kp are all rationals. Thus the straight line $y=\sqrt{2}\,x$ only passes through one lattice point, namely the origin.

This can be generalized to three dimensions. Thus we can speak of lattice points and point lattices in space. The equation $ax+by+cz=k$ represents a plane in a Cartesian coordinate system with the coordinates x,y,z. The problem of solving this equation in integers x,y,z is equivalent to the problem of determining all lattice points in space which lie on the plane. More generally, one may consider the distribution of lattice points on a given curve in the plane or on a given surface in space. This leads to the problem of solving Diophantine equations in two or three unknowns of any degree.

4.

PARTIAL FRACTIONS

In elementary arithmetic the problem of combining several fractions with different denominators is solved by converting the fractions into equivalent fractions with a common denominator. The reverse process of converting a fraction with a composite denominator into a combination of fractions with prime denominators or powers of prime denominators is possible with the aid of Euclid's algorithm. Consider, for example, the problem of decomposing the fraction 7/30 into such fractions (called partial fractions). We first factor 30 into two factors, say 5 and 6. Euclid's algorithm applied to these two numbers gives $1\times 6-1\times 5=1$. Hence, dividing both sides by 30 yields $1/5-1/6=1/30$. Treating the denominator 6 in the same way, we get $1/6=1/2-1/3$. Hence $1/30=1/5-1/2+1/3$, and

$$7/30 = 7/5 - 7/2 + 7/30 = 1\ 2/5 - 3\ 1/2 + 2\ 1/3 = 2/5 - 1/2 + 1/3.$$

The procedure is somewhat modified when the denominator contains factors which are powers of primes. Consider for example the decomposition of 4/45 into partial fractions. Euclid's algorithm applied to 9 and 5 gives $2\times 5-1\times 9=1$; dividing both sides by 45 yields $2/9-1/5=1/45$. Hence

$$4/45 = 8/9 - 4/5 = 2/9 + 2/3 - 4/5.$$

A similar process exists for polynomials in x with coefficients in any field. In High School Algebra the reader has undoubtedly learned how to divide one polynomial $n(x)$ by a second polynomial $d(x)$ so as to get a quotient $q(x)$ and a remainder $r(x)$ of degree less than that of the divisor $d(x)$. Although this process is usually carried out for polynomials with rational coefficients, it is not difficult to show that it is equally valid for polynomials with coefficients in any field. If we accept this result, then Euclid's algorithm described in detail for rational integers can be used to compute a polynomial $g(x)$ which is the g.c.d. of $n(x)$ and $d(x)$ and which is unique except for unit factors, i.e., non-zero elements of the coefficient field. It follows that $g(x)$ is expressible as a linear combination of $n(x)$ and $d(x)$:

$$p(x) \times n(x) + q(x) \times d(x) = g(x).$$

In particular, if the polynomials $n(x)$ and $d(x)$ are prime to each other, i.e., have no polynomial factor in common, then Euclid's algorithm assures us of the existence of two polynomials $p(x)$ and $q(x)$ such that

$$p(x) \times n(x) + q(x) \times d(x) = c$$

where c is a constant of the field independent of x.

As in the case with rational numbers, the last equation makes it possible to decompose rational functions of x into sums of simple fractions with denominators which are irreducible over the coefficient field, or powers of such polynomials. Although this result is usually derived in algebra by the method of "undetermined coefficients", it is necessary to use some other method such as our present method to prove that an expression in partial fractions always exists.

If the denominator $d(x)$ is a product of distinct and repeated factors such as $[d_1(x)]^{m_1}[d_2(x)]^{m_2} \cdots [d_k(x)]^{m_k}$ with integral exponents m, any two distinct irreducible $d_i(x)$ are prime and so are their powers $[d_i(x)]^{m_i}$. By the method described above, we may factor $d(x)$ in a manner so that one factor is $[d_i(x)]^{m_i}$ while the other factor is all the rest. Applying Euclid's algorithm to each such product will ultimately lead to the decomposition of $d(x)/n(x)$ into partial fractions.

Example. Consider the decomposition of $(x^2+1)/(x-1)^2(x+1)$ in the field of rationals. By the algorithm

$$1 \times (x-1)^2 - (x-3)(x+1) = 4.$$

Multiplying both sides by (x^2+1) and dividing both sides by $(x-1)^2 \times (x+1)$ yields:

$$4(x^2+1)/(x-1)^2(x+1) = (x^2+1)/(x+1) - (x^3-3x^2+x-3)/(x-1)^2.$$

Each of the fractions on the right may be simplified further by long division. Eventually, we derive the identity:

$$(x^2+1)/(x-1)^2(x+1) = 1/2(x+1) + (x+1)/2(x-1)^2 =$$

$$1/2(x+1) + 1/2(x-1) + 1/(x-1)^2.$$

5.

APPROXIMATION OF NUMBERS BY RATIONALS

In computation it is frequently convenient to replace complicated rationals or irrationals by rationals with small terms. For some purposes the fraction 1/7 is simpler than the corresponding repeating decimal .142587; the value of π is sometimes taken as approximately 22/7. Let us agree to call such approximations "simpler" than the original numbers. Clearly, it is desirable to study the best approximations of numbers by such simpler rationals. There will be a series of such approximations according to the degree of accuracy required. An elegant way to arrive at such approximations is by means of Euclid's algorithm as will be shown.

Suppose we start with a rational approximation p/q to a real number which is of greater accuracy than is required; p/q being in reduced terms. It is desired to find a simpler fraction p'/q' which will be an approximation to p/q. The difference $(p/q-p'/q')$ has a denominator which can be no greater than qq', and the absolute value of the difference cannot be less than $1/qq'$. Let us then call p'/q' a "best approximation" to p/q if $(p/q-p'/q') = \pm 1/qq'$. This is evidently equivalent to the equation:

$$q'p - p'q = \pm 1.$$

The numbers p', q' are precisely those obtained in Euclid's algorithm. Consecutive simpler approximations can be found in the same way from p'/q', and from the subsequent fractions obtained.

As an example, let us express 29/73 in a simpler form. Euclid's algorithm yields: $5 \times 29 - 2 \times 73 = -1$. Here $p=2$, $q=5$, and the fraction 2/5 is a best approximation to 29/73.

The process described in the previous paragraph is equivalent to the representation of rational numbers by finite continued fractions. The successive approximations are simply the corresponding convergents with p/q equal to the nth. convergent and p'/q' equal to the $(n-1)$st. convergent. To see this more clearly, let

$$p/q = a_1 + 1/a_2 + 1/a_3 + \cdots + 1/a_n \equiv [a_1, a_2, \cdots, a_n]$$

be the continued fraction expansion of p/q. If in this expansion, the last term $1/a_n$ is omitted, it can be shown[2] that the remainder of the

2. For this result and others quoted in this section, see Chapter X of G. H. Hardy's and E. M. Wright's "An Introduction to the Theory of Numbers", Oxford University Press, 1945.

fraction gives the first approximation p'/q'. More generally, it can be proven that for any consecutive pairs of convergents $p^{(r)}/q^{(r)}$, $p^{(r-1)}/q^{(r-1)}$

$$p^r q^{(r-1)} - p^{(r-1)} q^{(r)} = (-1)^{r-1}.$$

This is precisely the condition for Euclid's algorithm.

6.

EUCLID'S ALGORITHM IN THE RING OF GAUSSIAN INTEGERS[3]

As is well known the set of complex numbers $a+bi$ with a and b real constitutes a field. A Gaussian integer is defined when a and b are rational integers. Since $a+bi$ satisfies the equation

$$x^2 - 2ax + a^2 + b^2 = 0,$$

a Gaussian integer is an algebraic integer in the sense of the definition of such integers. We shall indicate the field of complex numbers over the of rationals by $R(i)$ and its ring by $R[i]$.

An integer U in $R[i]$ is said to be a "unit" of $R[i]$ if U is a divisor of G for every G in $R[i]$. Alternately, we may define a unit as any integer which is a divisor of 1. The two definitions are equivalent, since 1 is a divisor of every integer of the ring, and U dividing 1, 1 dividing G implies U divides G.

The "norm" of an integer G is defined by

$$N(G) = N(a+bi) = (a+bi)(a-bi) = a^2 + b^2.$$

If $\bar{G} \equiv a - bi$ is the "conjugate" of G, then $N(G) = G\bar{G} = |G|^2$. Since $(a^2+b^2)(c^2+d^2) = (ac-bd)^2 + (ad+bc)^2$, $N(GH) = N(G)N(H)$, i.e., the norm of a product is equal to the product of the norms.

In order to establish Euclid's algorithm for Gaussian integers, we shall need a construction analogous to Euclid's algorithm for rational integers. Such a construction depends upon the following:

3 For the benefit of the reader who is not familiar with the terminology used in this section, I state the following drfinitions:

A "ring" is a system of elements $a, b, c, \cdots$ which is closed under the two operations of addition and multiplication with the additional properties: I. $a+b=b+a$ II(i). $a+(b+c)=(a+b)+c$ II(ii). $a(bc)=(ab)c$ III. $a(b+c)=ab+ac$ IV. $a+0=a$ V. $a+(-a)=0$.

If in addition to these laws, $ab=ba$, the ring is said to be "commutative". Examples of commutative rings are offered by: (1) all rational integers, (2) all even integers, (3) all complex numbers.

A commutative ring which has the added property that if $ab=0$ then either $a=0$ or $b=0$ is called an "integral domain".

An integral domain in which division is possible by any element except zero is called a "field". The rational integers do not form a field; but the rationals, the real numbers, and the polynomials of elementary algebra each form a field.

Theorem 2. *Given any two Gaussian integers* G, G_1 $(G_1 \neq 0)$, *there exist two integers* G_2 *and* K *such that*

$$G = KG_1 + G_2, \quad N(G_2) < N(G_1).$$

Since $G_1 \neq 0$, we have $G/G_1 = r + si$ where r and s are rational. We can find two rational integers x and y such that

$$|r - x| \leq 1/2, \quad |s - y| \leq 1/2.$$

Hence

$$|G/G_1 - (x + iy)| = |(r - x) + i(s - y)| = [(r - x)^2 + (s - y)^2]^{1/2} \leq 1/\sqrt{2}.$$

Letting $K = x + iy$, $G_2 = G - KG_1$, we get

$$|G_2| = |G - KG_1| \leq 2^{-1/2}|G_1|.$$

Squaring both sides, we obtain

$$N(G) \leq 1/2\ N(G_1).$$

This proves the theorem.

If $G_2 \neq 0$, we apply Theorem 4 a second time, and we have

$$G_1 = K_1G_2 + G_3, \quad N(G_3) < N(G_2)$$

and so on. Hence, since the sequence of positive rational integers $N(G_1), N(G_2), \ldots$ is a decreasing one, the process must ultimately come to an end, and there must by an n for which

$$N(G_{n+1}) = 0, \quad G_{n+1} = 0,$$

and the last step of the algorithm will be

$$G_{n-1} = K_{n-1}G_n.$$

It now follows as in the case of rational integers that G_n is a common divisor of G and G_1, and that every common divisor of G_1 and G_2 is a divisor of G_n. Hence G_n is the greatest common divisor of G and G_1.

As in the case of the rational integers, the g.c.d. of two Gaussian integers is unique except for a unit factor. The units of the Gaussian integers are those numbers whose norms are 1. Hence all the units must satisfy the equation $a^2 + b^2 = 1$ for rational integers a and b; whose solutions are $a = \pm 1$, $b = 0$; $a = 0$, $b = \pm 1$. Hence the units are ± 1, $\pm i$.

7.

EUCLID'S ALGORITHM IN QUADRATIC RINGS

A complex number is called "algebraic" if it is the root of a polynomial with rational integer coefficients. In particular, it is an "algebraic integer" if it is the root of such a polynomial with leading coefficient 1. Hence a quadratic integer is a root of the equation

$$x^2 + px + q = 0$$

where p and q are rational integers. Let $d \equiv p^2 - 4q$, then a well-known result is the following:[4]

Theorem 3. *If* $d \neq 1$ *is an integer with no square factors, then in case* $d \equiv 2$ *or* $d \equiv 3$ (*mod* 4), *the algebraic integers of* $R(\sqrt{d})$ *are the numbers* $a + b\sqrt{d}$ *with* a *and* b *as coefficients. If* $d \not\equiv 1$ (*mod* 4), *the integers of* $R(\sqrt{d})$ *are the numbers* $a + b(1 + \sqrt{d})/2$, *with* a *and* b *rational integers.*

To study the possibility of constructing an Euclidian algorithm in the ring of quadratic integers it is advisable to use norms. Let $Q = a + b\sqrt{d}$ be a quadratic number, then its conjugate $\overline{Q} = a - b\sqrt{d}$ is also in the field, and the norm

$$N(Q) = Q\overline{Q} = (a + b\sqrt{d})\,(a - b\sqrt{d}) = a^2 - b^2 d$$

is a rational number. Since $(a^2 - db^2)(a'^2 - db'^2) = (aa' + bb'd)^2 - d(ab' + ba')^2$,

$$N(PQ) = N(P) \times N(Q),$$

and the norm transforms problems of divisibility and factorization of quadratic integers to that of rational integers. This is the reason why the process described for Gaussian integers in the previous section was effective in constructing an Euclidean algorithm for such integers. A similar remark applies for algebraic number rings of any degree, provided the norm of the elements of the ring have been properly defined.

Analogous to the case of Gaussian integers, let us suppose generally the proposition that given the quadratic integers Q and Q_1, there exists a quadratic integer K such that

7.1 $$Q = KQ_1 + Q_2, \quad |N(Q_2)| < |N(Q_1)|$$

is true in the ring $R[\sqrt{d}]$. This was proven valid for Gaussian integers;

4 See for example Garrett Birkhoff's and Saunders MacLane's "A Survey of Modern Algebra", The Macmillan Company, New York, 1941, p. 400.

but we have replaced $N(Q)$ by $|N(Q)|$ in order to include real rings ($d<0$). In these circumstances, we say that there exists an Euclidean algorithm in $R[\sqrt{d}]$, and call the ring "Euclidean".

If we write 7.1 in the form $Q_2/Q_1 = Q/Q_1 - K$, then the inequality in 7.1 is equivalent to the following: given any quadratic number A (integer or not) of $R(\sqrt{d})$, there exists an integer K such that

7.2 $$|N(A-K)| < 1.$$

Let $A = a + b\sqrt{d}$ where a and b are rational. If $d \not\equiv 1 \pmod 4$ then

$$K = x + y\sqrt{d}$$

where x and y are rational integers, and 7.2 is equivalent to

7.3 $$|(a-x)^2 - d(b-y)^2| < 1.$$

If $d \equiv 1 \pmod 4$, then

$$K = x + y(1+\sqrt{d})/2$$

where x and y are rational integers, and 7.2 becomes

7.4 $$|(a-x-y/2)^2 - d(b-y/2)^2| < 1.$$

When $d<0$ it is easy to determine all the rings in which inequalities 7.3 and 7.4 can be satisfied for any a,b and appropriate x,y. The results are stated in

Theorem 4. *There are just five complex Euclidean quadratic rings, namely the rings in which* $d = -1, -2, -3, -7, -11$.

When $d \not\equiv 1 \pmod 4$, we can choose rational integers x and y such that for given a and b, $|a-x| \le 1/2$, $|b-y| \le 1/2$, and by 7.3 require that $1/4 - d/4 < 1$. Since $d<0$, the only solutions possible are $d = -1, -2$.

When $d \equiv 1 \pmod 4$, we can for a given b, choose $|2b-y| \le 1/2$, and for a given a choose x so that $|a-x-y/2| \le 1/2$, and by 7.4 $1/4 - d/16 < 1$. Since $d<0$, the only solutions possible are $d = -3, -7, -11$. This completes the proof of theorem 4.

The situation is considerably more complicated for real quadratic rings. For the cases $d = 2,3,5,6,7,13,17,21,29$ the existence of Euclidean algorithms as been known for some time.[5] Only recently has it been proven that the number of real Euclidean quadratic rings is finite,[6] and the values of d for which this is true are:

5 For an elegant indirect proof of this result, see G. H. Hardy's and E. M. Wright's "An Introduction to the Theory of Numbers", p. 214.

6 Chatland, H. and Davenport, H., "Euclid's Algorithm in Real Quadratic Fields", Canadian Jour. of Math., Vol. 2, pp. 289-296, (1950).

$$d = 2,3,5,6,7,11,13,17,19,21,29,33,37,41,57,73,97.$$

Rings of a higher degree than the second have been proven Euclidean by a refinement of the technique used in quadratic rings.[7]

8.

EUCLID'S ALGORITHM AND UNIQUE FACTORIZATION

Once it has been established that a number ring possesses an Euclidean algorithm, it is then easy to prove that factorization is unique up to the unit factors in the integers of the ring. A "prime" of the ring is defined as a number of the ring which contains no other factors besides units of the ring and itself. Numbers which differ from one another by unit factors are called "associates".

Theorem 5. *Any integer, not zero, of an Euclidean ring can be expressed as a unit times a product of primes. This expression is unique except for the order in which the prime factors occur.*

In the case of the rational primes, this theorem is known as the "Fundamental Theorem of Arithmetic". We shall call rings in which unique factorization exists "simple". What Theorem 5 states then is that all Euclidean rings are simple.

To prove the theorem, we let p be a prime dividing $n = ab$. Then either p divides a or p divides b. For if p does not divide a, then by Euclid's algorithm there must exist integers p_1 and a_1 such that $p_1 p - a_1 a = 1$, and $p_1 pb - a_1 ab = b$. But p divides $p_1 pb$ and also $a_1(ab) = a_1 n$ so it must also divide b. The unique factorization now follows, for if p is one of the prime factors, in the first prime factorization it must divide one of the factors in the second factorization, i.e., must be an associate of the latter. Each factor in the first must be associated with a factor in the second, and uniqueness follows.

The theorem is not as trivial as it appears. It is by no means obvious that prime factorization should be unique. There are numerous examplex of rings where factorization of integers into primes is possible in different ways. Thus in the field $R(\sqrt{-5})$, the number 6 has the two distinct decompositions

$$6 = 2 \times 3 = (1 + \sqrt{-5}) \times (1 - \sqrt{-5}),$$

where the four numbers $2, 3, 1+\sqrt{-5}, 1-\sqrt{-5}$ can be shown to be distinct primes of the field.

That a ring may have unique factorization, it was shown above that it is sufficient that it possesses an Euclidean algorithm. But the condition is not necessary. Rings are known which do not possess algorithms and

7 See for example, Heilbronn, H., "On Euclid's Algorithm in Cyclic Fields", Canadian Jour. of Math., Vol. 3, pp. 257-268, (1951).

yet are simple. Such rings are the quadratic rings $R[\sqrt{-19}]$ and $R[\sqrt{-43}]$. This imposes a limitation on the usefulness of Euclid's algorithm to Number Theory. It is only with the introduction of Dedekind's "Theory of Ideals" that a deeper insight in the arithmetic of rings is obtained. In particular, the theory of ideals restores unique factorization in rings which are either simple or non-simple.

In developing Euclid's algorithm for quadratic rings, we made use of the norms of the field. This is not essential to the algorithm. The following definition is a generalization of the norm concept, and can be used effectively to establish an algorithm for the ring.[8]

Definition. *A commutative ring A with an identity is an Euclidran ring if there exists a function $f(a)$ defined in A and satisfying:* 1. *$f(a)$ is a non-negative integer, $f(a)=0$, if and only if $a=0$.* 2. $f(a\times b)=f(a)\times f(b)$. 3. *If $b\neq 0$, and a is arbitrary, then there exist elements q and r such that $a=b\times q+r$ where $f(r)<f(b)$.*

Another approach to the problem of finding the g.c.d. of two numbers in a ring and the proof of unique factorization in the ring depends upon the notion of a "module" of numbers. A module is a set S of numbers such that the sum and difference of any two numbers of S are themselves members of S. The numbers of a module need not necessarily be integers or even rational. But here we are concerned only with modules of integers. The single number 0 forms a module (the null module). It follows as a consequence of the definition of a module that

$$a-a=0,\quad a+a=2\times a,\quad a+a+a=3\times a,\ \cdots,\ a+a+a+\cdots+a=n\times a$$

all belong to S if a belongs to S. More generally, $m\times a+n\times b$ belongs to S for any rational integers m,n and any members a,b in S.

In what follows, we limit ourselves to modules of rational integers. Much of the theory carries over to modules of other number systems as well. It is plain that any module S of rational integers, except the null module, contains some positive number. Suppose that d is the smallest positive number of S. If c is any positive number of S, then $(c-n\times d)$ belongs to S for all n. If r is the remainder when c is divided by d, and $c=n\times d+r$, then r belongs to S and $0\leq r<d$. Thus $r=0$, and $c=n\times d$. Hence

Theorem 6. *Any module, other than the null module, is the aggregate of integral multiples of a positive number d.*

We next show that the number d defined above is the g.c.d. of a and b, i.e., $d=(a,b)$. Let $d=m\times a+n\times b$. Then any common factor of a and b must be a factor of d. But, the integers $a=1\times a+0\times b$ and $b=0\times a+1\times b$ both lie in the module under consideration, and hence must be multiples of the minimum number d in the set. Hence d is the g.c.d. of a and b.

We have thus shown the existence of the g.c.d. of two numbers by the theory of modules. There is a vital difference when compared to the process of deriving the g.c.d. by Euclid's algorithm: Euclid's algorithm offers a direct construction for the g.c.d. of two integers; while the theory of modules merely proves the existence of such a g.c.d.

8 See for example, Nathan Jacobson's "Lectures in Abstract Algebra", D. Van Nostrand Co., Vol. 1, p. 122.

SO YOU THINK YOU CAN COUNT!

Cpl. Jerry Adler

Of course you can count. Like most Americans, you had schooling in arithmetic. As a matter of fact, you may have mastered the slide rule, enabling you to do mathematics like a whiz.

Brother, are you in for a surprise.

In Japan they have an arithmetic board called the soroban. Believe it or not, that little calculator outdoes anything we've got.

Ridiculous? Impossible? Well, let's take an example or two.

In November, 1946, a contest was held between the Japanese abacus and the electric calculating machine in Tokyo under the sponsorship of the service newspaper, Stars and Stripes.

An American Army private, selected in an arithmetic contest as the most expert operator of the electric calculator in Japan, met Kiyoshi Matsuzaki, champion operator of the abacus in the Ministry of Postal Administration, on the field of battle.

The abacus won four to one — going away .

Still skeptical? Want more proof?

Stars and Stripes remarked, in reporting the contest, "The machine age took a step backward yesterday at the Ernie Pyle Theater as the abacus, centuries old, dealt defeat to the most up-to-date electric machine now being used by the U.S. Government."

Still not satisfied?

The following account should suffice to dispel all doubt.

In May 1952, during the Sixth All-Japan Abacus Contest, held in Tokyo, a master abacus operator gave a demonstration of skill in mental arithmetic. This method consists in mentally visualizing a soroban and working a problem out by standard methods on the imaginary instrument.

In less than one-and-one-half minutes he gave correct anseers to 50 division problems, each containing five to seven digits in dividend and divisor. Then, in less than 14 seconds he added ten numbers of ten digits each.

All is not Lost

There is one consolation. Some of us in Japan are learning how to use the soroban.

Under the guidance of Patricia Denslow (Pasadena, Calif.), 4th grade teacher at the Negishi Heights Dependent's Elementary School near Yokohama, youngsters are learning the mysteries of the amazing abacus.

Miss Denslow's interest in the abacus began while at the University of California at Los Angeles (UCLA) where she majored in psychology.

Research in mathematics led to data on the abacus influencing her to make the teaching of arithmetic a life's work.

To learn more about the abacus, Miss Denslow took a year's leave of absence from the schools of Manhattan Beach, California, and accepted a position with Japan dependent schools.

Utilizing her own publication, "The Soroban Made Easy", Miss Denslow has uncovered a host of valuable assets on the ancient instrument.

It helps break the bugaboo of counting on fingers, a device which dogs the progress of many children. It reduces tension by giving activity to the fingers. It requires the habit of mental calculation; essential to speed and confidence in paper work.

Experience with the soroban forces the child to deal concretely with numbers and to observe absolutely essential aspects of our number system. And children who have trouble with written work in the first three grades, seem to catch on for the first time when soroban studies begin in the fourth grade. Thus the experience carries over to written work.

The soroban is one of the first cultural objects which attracts attention in Japan, thus it acts as a magnet on the childs interest.

Children go on shopping trips with their parents, and are both perplexed and stimulated by seeing a tradesman reach for his soroban.

He tilts the calculator, sweeps his index finger accross the surface, pushes little beads one way and another, and announces a price.

Not only children, but most Westerners are interested in the primitive looking instrument which makes arithmetic so painless.

Reckoning Table Covered with Dust

An immediate assumption is that the soroban must have been invented by a renowned mathematician. Was it the Greek Pythagoras, famous for the theorem "in a right triangle the square of the hypotenuse equals the sum of the squares on the other two sides"? Could it have been ancient Euclid, renowned author of the geometric treatise "Elements"?

Probably no one will ever know the answer.

We do know certain facts however. The English word abacus is etymologically derived from the Greek abax, defined as a reckoning table covered with dust or sand. As a result, it is believed that the abacus, in its earliest form, was a 'reckoning table covered with dust', in which figures were drawn with a stylus.

In time, a ruled table with disks on lines to indicate numbers, came into being in numerous forms. All were found at one period or another in ancient Rome.

From one variation, the grooved abacus, a type with beads or rods, the soroban was probably developed.

The Japanese soroban, thought to be a rendering of the Chinese suanpan, has undergone many improvements since its popular inception in Japan back in the sixteen-hundreds.

Today the soroban has been simplified into an oblong wooden frame, about a foot long and two inches wide. The frame holds some 21 vertically arranged rods, on which wooden beads or counters slide up and down. a beam, about 1/4 of the way from the top, slices across the frame horizontally.

Each of the four beads on the lower section of a rod has the value of one; the bead above the crosspiece has a five value. When all beads are touching the frame the board equals zero. Each of the one unit beads obtain value when moved up against the crosspiece and loses value when moved back down to its former position. Each of the five unit beads above the crosspiece obtains value when moved down against the crossbeam.

When all beads equal zero, accomplished by moving all fives beads up and all ones beads down, the board is neutral.

Starting at a dot on the crosspiece, placed on every fourth rod, one is ready to begin working the soroban.

The beads rod at the chosen dot column is designated the digits column. To the left of that rod is the tens column. Next is the hundreds column and so forth. To the right of the chosen digits column is the tenths column.

The arithmetic method has a tremendous advantage in addition and subtraction. A problem is worked out from left to right, rather than from right to left, as in the case of written arithmetic. It is therefore possible to add or subtract while numbers are being given. For example, if the first number in a problem is 652, the operator can enter 6 on the abacus the instant he hears or sees six hundred. He proceeds to the 5 and finally the 2. In addition he would add in the hundreds column first and not wait until all figures were given to start backward from right to left.

Other advantages of the abacus are its speed of calculation, its moderate price, its portability, and its handy construction.

One disadvantage of the abacus is that intermediate steps in calculation are not preserved since only a final result is produced. An error would therefore necessitate the complete recalculation of a problem.

Yet, can one doubt the value of a 2,000 year-old instrument which has proven itself superior to the electric calculating machine?

How Much is 582 Plus 88?

Clear the soroban by pushing all beads toward the frame. A designated dot on the crosspiece is the digits column. 'Put on' a fives bead in the hundreds column by sliding the bead above the crosspiece down. Continuing to the right, move the fives bead in the tens column down and three ones beads up to get 80. Finally, move two beads in the ones column up to get two.

Remember that in addition you must think in terms of ten. Remember that there are nine beads on each vertical rod. To get 10 you need a ones bead moved up to the horizontal crossbar in the tens column. This is necessary because the bead in the digits column moved down to the

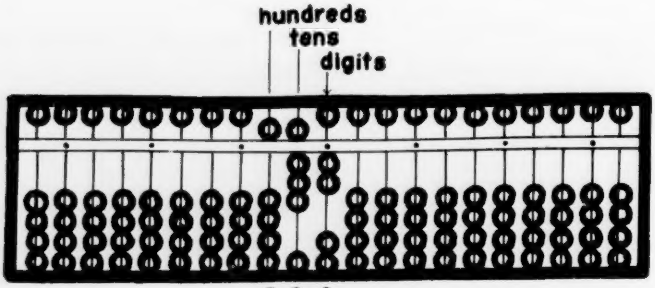
crosspiece (equal to five) plus four ones beads in the digits column moved up to the crossbar (equal to four) equals only nine.

To add 88 to 582 already 'set' on the soroban, we proceed from right to left.

We leave the fives bead in the hundreds column alone. In the tens column we want to add eighty. Remembering to figure in terms of ten, we *subtract* eight from 10. The resulting two makes us remove two beads in the tens column and to compensate on the board we add one bead (below the crossbar) in the hundreds column. In other words, we take away 20 in the tens column and add one bead below the crossbar in the hundreds column, leaving 80 in the tens column.

To add eight in the digits column we again subtract from ten. Removing two beads in the ones column, we again add one bead (below the crossbar) in the tens column to balance the board.

We have the answer – 670!



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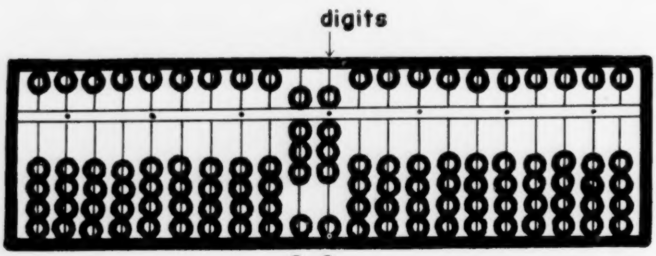

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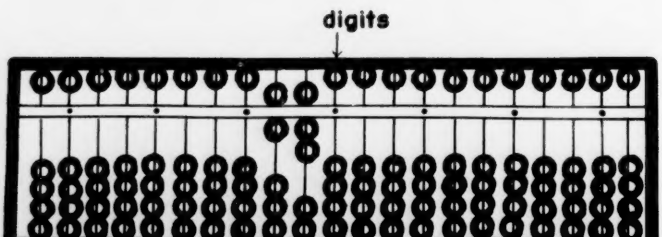


6 7 0

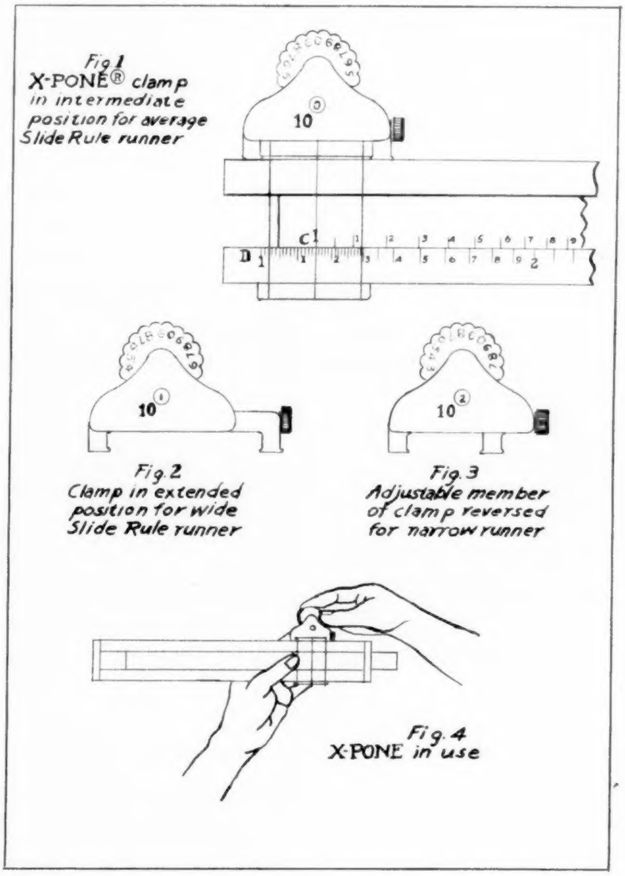
A MECHANICAL DEVICE, "THE X-PONE", FOR POINTING OFF DECIMALS*

C. L. Weckesser

The **X-PONE**** is an instrument which, when used in combination with a slide rule, enables one to read *actual* numbers in slide rule computations.

Attach the X-PONE to the runner or indicator of your slide rule as shown in Figs. 1, 2 or 3.

At the start of any slide rule computation, always set the dial of



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the X-PONE so that the blue $(^0)$ shows at the window or viewing aperture. At this setting the X-PONE is read 10^0 (ten to the zero power).

The little number at the upper right of the 10 is called an exponent and represents the number of times 10 must be multiplied into 1. For example:

$$10^0 = 1;$$

$$10^1 = 10;$$

$$10^2 = 10 \times 10 = 100; \text{ similarly}$$

$$10^3 = 1{,}000.$$

If this little number is negative it represents the number of times 1 is divided by 10. For example:

$$10^{-1} = \frac{1}{10} = .1$$

$$10^{-2} = \frac{1}{10 \times 10} = .01$$

The blue numbers on the X-PONE dial represent (+) exponents; the red numbers (-) exponents, as follows:

Blue	*Red*
$(^0) = 10^0 = 1$	$(^1) = 10^{-1} = .1$
$(^1) = 10^{+1} = 10$	$(^2) = 10^{-2} = .01$
$(^2) = 10^{+2} = 100$	$(^3) = 10^{-3} = .001$
$(^3) = 10^{+3} = 1000$	$(^4) = 10^{-4} = .0001$
etc.	etc.

To multiply powers of 10, it is only necessary to add exponents, thus

$$100 \times 1{,}000 = 10^2 \times 10^3 = 10^{2+3} = 10^5 = 100{,}000;$$

$$100 \times .001 = 10^2 \times 10^{-3} = 10^{2+(-3)} = 10^{-1} = .1$$

In division, exponents are subtracted, thus:

$$\frac{100}{1{,}000} = \frac{10^2}{10^3} = 10^{2-3} = 10^{-1} = .1$$

and

$$\frac{100}{.001} = \frac{10^2}{10^{-3}} = 10^{2-(-3)} = 10^{2+3} = 10^5 = 100{,}000$$

Addition or subtraction of exponents can be very easily done mechanically on the X-PONE. By turning the dial clockwise and counting the "clicks" exponents are added. By turning the dial counter clockwise exponents are subtracted. This can be done without the use of paper and pencil. Consider:

$$\frac{100}{10{,}000}$$

Look at the 100 and turn the dial two clicks clockwise. A blue (^{2}) should show in the window of the X-PONE. Next look at the 10,000 in the denominator and turn the dial counter clockwise four clicks. A red (^{2}) will now show in the window. The answer is 10^{-2} or .01.

In such a case as:

$$\frac{100}{.0001}$$

after setting the dial to a blue (^{2}) for the 100 look at the .0001 and, remembering that $-(-) = +$, turn the dial four more "clicks". A blue (^{6}) will show in the window.

The real value of the X-PONE is apparent when dealing with combined multiplication and division of numbers. Consider:

$$\frac{1{,}000{,}000 \times 1{,}000 \times .00001}{100{,}000 \times .0001}$$

Starting with a blue (^{0}) count for the numerator clockwise (^{6}), clockwise (^{3}) and counter clockwise (^{5}). For the denominator count counter clockwise (^{5}) and clockwise (^{4}). Read the answer 10^3 or 1,000.

In using the X-PONE in combination with the slide rule, the *C* and *D* scales of the slide rule are taken to read from 1 to 10. All numbers set on, or read from the slide rule are considered as units multiplied by powers of 10. For example, with the runner set on the *D* scale as shown in Fig. 1, the X-PONE slide rule combination reads

$$1.15 \times 10^0 = 1.15 \times 1 \quad \text{or } 1.15$$

If the dial of the X-PONE is now turned clockwise so that a blue (^{1}) shows in the window, the reading is 1.5×10^1 or 11.5. If the dial is next turned counter clockwise two "clicks" so that a red (^{1}) shows in the window, the reading is 1.15×10^{-1} or .115.

Multiplying, for example, $.00115 \times 625$ is the same as $1.15 \times 10^{-3} \times 6.25 \times 10^2$. The result is 7.19×10^{-1} or .719, while dividing $\frac{.719}{625} = \frac{7.19}{6.25} \times \frac{10^{-1}}{10^2} = 1.15 \times 10^{-3}$ or .00115.

If the product of two units, multiplied on the *C* and *D* scales, exceeds 10, the slide or *C* scale will extend to the left of the slide rule. For example: $3 \times 5 = 15$. In this case the dial of the X-PONE must be turned one "click" to add (^{1}) to the exponent of 10. The result is then read 1.5×10^1 or 15.

If a smaller unit is divided by a larger unit, the slide will project to the left as in 3/5. In this case subtract 1 from the exponent of 10 by turning the X-PONE dial one "click" counter clockwise. Read the answer, 6×10^{-1} or .6.

In combined multiplication and division, as in the following example:

$$\frac{576 \times .000354 \times 100,000}{.0000076 \times 92,000}$$

several procedures can be followed: (1) The fastest procedure is (a) Total up the powers of 10 on the X-PONE, (b) Compute the multiplication and division of the units on the slide rule, making changes in the X-PONE *only* when the slide projects to the left. The steps would be as follows:

(a) Collecting, on the X-PONE, the powers of 10 arising from reducing the numbers to units,

$$\frac{\text{numerator}}{(^2) - (^4) + (^5)} \qquad \frac{\text{denominator}}{+ (^6) - (^4)}$$

(b) Manipulating the slide rule to perform the successive steps of division, multiplication, and division,

$$\frac{5.76}{7.6} \times \frac{3.54}{9.2},$$

additional exponents arise, indicated by the slide extending to the left. These are also set in the X-PONE, $-(^1) + (^1) - (^1)$. Hence the final result is 2.92 on the slide rule and 10^4 on the X-PONE, and is read 2.92×10^4 or 29,200.

(2) The other procedure is to do the X-PONE operations stepwise, along with the slide rule operations, thus

(a) $\dfrac{576}{.0000076}$ reads 7.59 on the slide rule and 10^7 on the X-PONE.

(b) $75,900,000 \times .000354$ reads 2.68 on slide rule and 10^4 on X-PONE.

(c) $\dfrac{26,800}{92,000}$ reads 2.92 on the slide rule and 10^{-1} on the X-PONE.

(d) $.292 \times 100,000$ reads 2.92 on the slide rule and 10^4 on the X-PONE.

Note that with procedure (2) the decimals can be pointed off, if desired at each step in the computation.

In using the X-PONE slide rule combination for scales other than *D* and *C*, the operation of the X-PONE is the same for the powers of 10. However, the following notes apply to the operation of the slide rule and its effects on the X-PONE:

I *D and C Scales.* If the slide projects to the *left*, add $(^1)$ on the X-PONE in multiplication, subtract $(^1)$ in division.

II *D and CI Scales.* If the slide projects to the *right*, add $(^1)$ on the X-PONE in multiplication, subtract $(^1)$ in division.

III *D, CF and DF Scales*. If the slide projects to the *left* in multiplication and the runner is to the *left* of the *CF index* or to the right of the *DF index*, add (^{1}) on the X-PONE. If the slide projects to the *right* in multiplication and the runner is between the *CF* and *DF* indices, add (^{1}) on the X-PONE. If slide projects to the *left* in division subtract (^{1}) on the X-PONE.

IV *D, CIF and DF Scales.* If the slide projects to the *right* in multiplication add (^{1}) on the X-PONE. If the slide projects to the *right* in division and the runner is to the *left* of the *CI index* or to the *right* of the *DF index*, subtract (^{1}) on the X-PONE. If the slide projects to the *left* in division and the runner is *between* the *CIF and DF indices*, subtract (^{1}) on the X-PONE.

V *A Scales*. The left hand *A* scale is taken to run from 1 to 10, the right hand scale from 10 to 100.

To square a number (a) Consider the number as a unit multiplied by the proper power of 10. (b) Double the power of 10 and set on the X-PONE. (c) Set the runner of the slide rule to the unit on the *D* scale and read the answer on the corresponding *A* scale. Examples:

(1) $(310)^2 = (3.1)^2 \times 10^4$. The answer will be found on the left hand *A* scale as 9.6×10^4 or 96,000.

(2) $(.0332)^2 = (3.32)^2 \times 10^{-4}$. The answer will be $11 \times 10^{-4} = 1.1 \times 10^{-3}$ or .0011.

To extract a square root of a number: (a) Move the decimal point an even number of places to the right or left to obtain a whole number between 1 and 100. (b) Set ½ the number of places moved in step (a) on the X-PONE according to sign, i.e. (+) if the decimal place was moved to the left, (-) if moved to the right. (c) If the whole number obtained in step (a) is between 1 and 10 set the runner on the left hand *A* scale or on the right hand *A* scale if the number is between 10 and 100. Find the square root of the number on *D*. Point off the answer as indicated on the X-PONE. Examples:

(1) $\sqrt{62,500} = \sqrt{6.2 \times 10^4} = \sqrt{6.25} \times 10^2 = 2.5 \times 10^2$ or 250.

(2) $\sqrt{.0005} = \sqrt{5 \times 10^{-4}} = \sqrt{5} \times 10^{-2} = 2.24 \times 10^{-2}$ or .0224.

VI *K Scales*. The left hand *K* scale is taken to run from 1 to 10, the middle *K* scale from 10 to 100, the right hand *K* scale from 100 to 1,000.

To cube a number: (a) Consider the number as a unit multiplied by the proper power of 10. (b) Multiply the power of 10 by 3 and set it on the X-PONE. (c) Set the slide rule runner on the *D* scale and read the cube of the unit on the corresponding *K* scale. Examples:

(1) $(210)^3 = (2.1 \times 10^2)^3 = 9.26 \times 10^6$ or 9,260,000.

(2) $(.035)^3 = (3.5 \times 10^{-2})^3 = 43 \times 10^{-6} = 4.3 \times 10^{-5}$ or .000043.

To extract the cube root: (a) Move the decimal point to the right or left a number of places, divisible by 3, to obtain a whole number between 1 and 1,000. (b) Set 1/3 of the number of decimal places moved, on the X-PONE according to sign, (+) or (–). (c) Set the number obtained in step (a) on the proper K scale, 1 to 10, 10 to 100 or 100 to 1,000. Read the cube root on the D scale and point off as indicated on the X-PONE. Examples:

(1) $\sqrt[3]{8250} = \sqrt[3]{8.2 \times 10^3} = \sqrt[3]{8.2} \times 10^1 = 20.2$

(2) $\sqrt[3]{.000027} = \sqrt[3]{27 \times 10^{-6}} = .03$

VII *T, S and ST Scales*. If at the end of a computation for tan θ, the X-PONE reads 10^{-1}, the angle θ will be found on the T scale. If the X-PONE reads 10^{-2}, the angle θ will be found on the ST scale. If the X-PONE reads 10^0 or 10^1, the angle $\theta = 90 - \text{arc tan } (\frac{1}{\tan\theta})$. Examples:

(1) Arc tan $5.77 \times 10^{-1} = 30$ degrees (found on T).

(2) Arc tan $1.91 \times 10^1 = 90 - \text{arc tan } (5.24 \times 10^{-2}) = 90 - 3$ (found on ST scale) = 87 degrees.

If the X-PONE reads 10^{-1} for sin θ, the angle θ will be found on the S scale. If the X-PONE reads 10^{-2} for sin θ, the angle θ will be found on the ST scale.

VIII *LL1, LL2, LL3, LLO1, LLO2, and LLO3 Scales*. The X-PONE can be used to determine the correct LL scale on which the answer will be found. Set the X-PONE to the blue number corresponding to the number LL or LLO scale on which the first quantity is set. Then proceed in a manner similar to multiplication, since

$$\log_{10}(\log_e X^n) = \log_{10} n + \log_{10}(\log_e X)$$

Examples:

(1) Evaluate $8^{0.5}$. Set the right index on the C scale opposite 8 on the $LL3$ scale. Set the X-PONE to $(^{+3})$. Move the runner to 5 on the C scale. Subtract $(^1)$ on the X-PONE for $0.5 = 5 \times 10^{-1}$ and then add $(^1)$ on the X-PONE since the slide of the rule projects to the left. This leaves the X-PONE set at $(^{+3})$. Therefore, the answer is found on the $LL3$ scale as 2.83.

(2) $8^{0.005} = 1.0105$ (read on $LL1$ scale) since the X-PONE will read $(^{+1})$ after manipulations.

THE INFLUENCE OF NEWTONIAN MATHEMATICS ON LITERATURE AND AESTHETICS

Morris Kline

All Nature is but Art, unknown to thee;
All Chance, Direction, which thou canst not see;
All Discord, Harmony not understood.

Alexander Pope

During his travels in Laputa Gulliver encountered several professors engaged in projects to improve the language of the country. The first project was to shorten discourse by cutting polysyllables into one, and leaving out verbs and participles, because in reality all things imaginable are but nouns. Another project was a scheme for abolishing all words whatsoever; this was urged as a great advantage in point of health as well as brevity. Words could be dispensed with simply by having people carry objects about with them and exhibit these instead. Unforttunately the women of Laputa objected to this scheme because it did not allow them to use their tongues.

In this passage as well as numerous others Jonathan Swift used his strongest weapon-satire—to ridicule and to nullify the thoroughgoing influence on literature exercised by the mathematics of his day. Just as the successful business man in twentieth century America has become the authority in our time, somathematics, successful in revealing and phrasing the order in nature, became the arbiter of language, style, and form in the literature of the seventeenth and eighteenth centuries.

The imitation of mathematical writings was a major influence towards standardizing language. Words became arbitrary but fixed symbols for ideas just as x is an arbitrary symbol for an unknown quantity and $y = x^2$ for a functional relation. The standardization of the English language could be noted in the constant reference to girls as nymphs, lovers as swains, lawns as dewy, fountains and streams as mossy, and water as limpid, while particular words were associated *ad nauseam*.

Just as mathematics uses abstract concepts so did ordinary discourse. A gun became a leveled tube; birds were a plumy band; fish were a scaly breed or a finny race; the ocean became a watery plain; and the sky a vault of azure. The poets in particular indulged in abstract terms such as virtue, folly, joy, prosperity, melancholy, horror and poverty, which they personified and wrote in capital letters. Both standardization and the preference for abstractness stripped the language of concrete, colorful, picturesque, and succulent words.

The movement towards standardization culminated in one of the landmarks of the development of the English language, Samuel Johnson's

Dictionary. Johnson undertook to regulate a language which had been "produced by necessity and enlarged by accident." From a more or less inclusive explanation of the meanings of words Johnson converted the dictionary into an authoritative standard of good usage and the arbiter of verbal fashions. By reference to actual quotations and by careful distinctions clearly set forth he established exact meanings and proper employment of words. It was his intent that these meanings and usages were to be fixed for all time just as the word "triangle" has meant precisely the same thing for 2500 years.

This change in the concept of a dictionary appears radical in a history of dictionaries but it was almost to be expected in the eighteenth century. For Johnson set about to do for the English language what the age had already decided to do in all spheres of activity, namely, to determine and establish the most reasonable, most efficient, and permanent standards. Actually Dr. Johnson was misguided. Philologists have learned since his day that language is necessarily a fluid, slowly changing, evolving phenomenon. Words change in meaning from year to year and from place to place. In recognition of changes in language the modern dictionary reserves a place for archaic meanings.

Standardization of language was accompanied by a critical examination of the efficacy of ordinary languages. Jeremy Bentham, who is noted for his highly influential political philosophy, concerned himself also with the achievement of precision in language. Nouns, he says, are better than verbs. An idea embodied in a noun is "stationed on a rock"; one embodied in a verb "slips through your fingers like an eel". The ideal language would resemble algebra. Ideas would be represented by symbols as numbers are represented by letters. These ideas would be associated by the smallest possible number of syntactical relationships just as all numbers are associated by just the few operations, addition, multiplication, equality, and so forth. Two statements should then be comparable as are two equations when, for example, one equation is obtained from another by multiplication by a constant.

The influences of mathematics on language which we have examined thus far were in the nature of modeling ordinary language on the pattern of mathematics. The mathematician and philosopher Leibniz attacked the problem of language from the other end. Mathematics already had an ideal language and modes of discourse for its purposes. Why not broaden the scope of the mathematical language and mathematical machinery so as to include all domains of inquiry? His first projected step towards this universal science or logical calculus was to decompose all the ideas which are employed in human thought into the fundamental, distinct ones, just as the numbers 6 and 24 are decomposable into the ultimate prime factors 2 and 3. He next proposed to represent these fundamental ideas by symbols of a universal, scientific language, symbols not unlike Chinese ideogram writing. The symbols would be compact, unambiguous, and non-overlapping. He then intended to codify the laws of reasoning so

that one could apply them to the symbols and combinations of symbols in order to deduce conclusions mechanically and efficiently, just as mathematics does in algebra. By this plan he hoped that *all* questions could be settled.

At first blush Leibniz' plan seems preposterous. To hope to settle all questions in all fields strikes one as rather far-fetched. Yet this much can be said in his behalf. The history of mathematics shows that the introduction of better and better symbolism has made a commonplace of processes which would have been impossible with the unimproved symbols. To take the simplest example, the use of the Hindu-Arabic symbols for our numbers and of positional notation makes it possible for elementary school children to-day to perform operations beyond the capacities of learned mathematics of Greek, Roman, and medieval times. Nevertheless Leibniz was too ambitious. Not only did he himself never complete his efforts in these directions, but his belief that all ideas could be decomposed into relatively few fundamental ones has not been substantiated.

The reform of language was but one of many mathematical influences on literature. Style was radically altered. It was well known that statements in a mathematical discussion or demonstration are concise, unambiguous, clear, and precise. Many felt that the success enjoyed by mathematics could be credited almost entirely to this unquestionably naked and pristine style. While this view falls short of the truth, it nevertheless reflected the opinions of the educated public. One of the great intellectuals of the age, Le Bovier de Fontenelle, wrote:

> The order, the neatness, the precision, the exactness prevailing in good books for some time may well have arisen in that geometrical spirit now more widespread than ever ···. A work on ethics, politics, or criticism, perhaps even a work of eloquence will be finer, other things being equal, if it is done by the hand of a geometrician.

Dryden went so far as to declare: "A man should be learned in several sciences, and should have a reasonable, philosophical and in some measure, a mathematical head to be a complete and excellent poet ··· ."

Young America also fell under the new influences.

> We do not listen with the best regard to the verses of a man who is only a poet, nor to his problems if he is only an algebraist; but if a man is at once acquainted with the geometric foundation of things and with their festal splendor, his poetry is exact and his arithmetic musical.

This from Emerson.

The works of the outstanding mathematicians were set up as literary models by the eighteenth century. Descartes' style was extolled for its clarity, neatness, readability, and perspicuity, and Cartesianism became a style as well as a philosophy. The elegance and rationality of Pascal's manner, especially in his *Lettres Provinciales*, were hailed as superb elements of literary style. Writers in almost all fields began to ape as closely as their subject matter permitted the works of Descartes, Pascal, Huygens, Galileo, and Newton.

The changes in prose style were numerous. Abandoned was the pedantic, florid, scholarly style with complex Latinized constructiins in favor of a simple, more direct prose. Banished also were impetuous flights of imagination, vigorous, emotionally charged expressions, poetic exuberance, enthusiasm, and sonorous and highly suggestive phrases. An age of reason demanded a prose which would be readily intelligible and appeal to the common sense of men.

In the seventeenth century the Fellows of the Royal Society in London decided that the reformation of English prose was within the province of that august body. A committee, including Sprat, Waller, Dryden and Evelyn, was appointed for the study of the language. With furtive glances at the Académie Francaise the committee suggested founding an English academy for the "improvement of speaking and writing." It also urged the members of the Society to avoid eloquence and extravagance of expression in the description of their experiments. They were to reject all "amplifications, digressions and swellings of style" and to seek a "return to primitive purity, and shortness, when men delivered so many things in almost an equal number of words." They were to use a "close, naked, natural way of speaking; positive expressions, clear senses; a native easiness; *bringing all things as near the mathematical plainess as they can;* and preferring the language of artisans, countrymen, and merchants before that of wits and scholars."

And so a style based on reason and order evolved. Clarity, proportion, the architectural instinct for form, the gift of logic, rhythm, symmetrical structure and cadences, and rigid adherence to set forms were elements of the new prose style. Prose became sober, terse, precise, and epigrammatic, largely in imitation of the aphoristic, sententious manner of Francis Bacon. A demand for easy intelligibility and clarity required that each phrase or group of words be readily grasped by the mind. Hence brief sentences became fashionable. Inversion was frowned upon; within the sentence the order of words was dictated by the thought. Also, sentences were supposed to link with each other so as to show clearly and at once whence the thought comes and where it goes. The aim and law of prose style became the "easy intelligible intercourse of minds".

This emphasis on the rational and scientific elements in style at the expense of the emotional fostered the qualities of perspicuity and logic appropriate to fine rhetoric, reasoning and narrative, and discouraged the expression of strong emotion and passion that inspires great poetry. Literary pleasure was sacrificed to rational demands. The Age of Reason expressed itself, therefore, most characteristically in prose, the novel, diary, letter, journal and essay lording it over the lyric and drama. In fact the novel pretty much replaced poetry as the outlet for imaginative writing while the lyric poetry of the age became prosaic, "poetized prose."

This change wrought by Newtonian science and mathematics parallels to a large extent the change ushered in by Socrates in Greece of the fourth century B.C. In the words of the renowned Greek historian, Bury:

> While Socrates and others had been bringing about a revolution in thought, the ··· professors of rhetoric or style had been preparing an efficient vehicle for diffusing ideas. Prose is the natural instrument of criticism and argument; it is a necessary weapon for intellectual persuasion; and therefore the fourth century is an age of prose.

The parallel between the literature of the European and Greek ages of reason goes even further. The same drift from religion characterized both the eighteenth century and the fourth century B.C. and affected in a similar fashion their respective literatures. Let us read Bury again:

> It must always be remembered that the great dramatic poets of the fifth century bore an inalienable religious character; and, as soon as the day came when the men of the highest literary faculty were no longer in touch with the received religion, drama of the old kind ceased to be written.

Not content with reducing poetry to a minor activity, the critics of the eighteenth century were determined to achieve mathematical objectivity by suppressing all personal or individualistic efforts in this art. Pope, Addison, and Johnson literally dictated poetic style in accordance with strict rules derived from a study of the ancients, while Dryden's translations of the Latin classics prrscribed the laws of metrical tranlation into English. Verse, it was thought, could be taught by rule; lyric, epic, sonnet, epistle, didactic verse, ode and epigram could be built up by observance of their natural laws, while order, lucidity and balance were goals sought in the process. Attention to grammatical rules and sentence structure was recommended for verse. The principles of form in poetry were likened to mathematical axioms because the axioms determined the form as well as the content of the theorems. The heroic couplet won favor because of its balance and symmetry, and, extreme as it may seem to us, because the form was analogous to a series of equal proportions. It was regarded as the essence of cadenced regularity.

Poetry adopted a code which was laid down as a series of mathematical propositions. Great poetry reduced to correct writing, that is, obedience to the code. In this code, spirit as well as form were prescribed; enthusiasm was abhorrent; there was to be no emotion, no abandon. Imagination was to be limited by reason. The goal of poetry was not to stir the feelings, but to be didactic, ratiocinative and argumentative in rhyme.

Pope's own expression of the belief in "natural laws" for poetry is found in his *Essay on Criticism*.

> First follow Nature and your judgment frame
> By her just standard, which is still the same.
> Unerring Nature, still divinely bright,
> One clear, unchanged and universal light,
> Life, force, and beauty, must to all impart,
> At once the source, and end, and test of Art.

To follow nature did not mean precisely what it meant in the physical sciences, that is, to obey nature's mathematical laws. Rather through the historically justifiable association of the Greeks with nature, to follow her meant to imitate the form of the Greek classics. Hence, says Pope:

> Those rules of old discovered, not devised,
> Are nature still, but nature methodized;
> Nature, like liberty, is but restrained
> By the same laws which first herself ordained.
> ....
> When first young Maro [Virgil] in his boundless mind
> A work t'outlast immortal Rome designed,
> Perhaps he seemed above the critic's law,
> And but from Nature's fountains scorned to draw;
> But when t'examine every part he came,
> Nature and Homer were, he found, the same.

Nevertheless when Pope translated Homer's *Iliad* he rendered not Homer but Pope. As Leslie Stephen points out in *English Literature and Society in the Eighteenth Century* "When we read in a speech of Agamemnon exhorting the Greeks to abandon the siege

> Love, duty, safety summon us away;
> 'Tis Nature's voice, and Nature we obey,

we hardly require to be told that we are not listening to Homer's Agamemnon but to an Agamemnon in a full-bottomed wig." We need hardly be told also that we are listening to the voice of the eighteenth century attuned to its basic assumptions: the validity of rationalism and the prevalence of natural law.

As a consequence of the pressure of the critics in the directions indicated poetry did become decorous, temperate, well regulated, and intellectual. The rules of the critics were followed almost meticulously. Poetry adopted Pope's formal and strictly regulated versification, and emphasized such neo-classical ideals as lucidity, moderation, elegance, proportion, and universality. Poetry could be ironic but must suppress feeling and, especially, enthusiasm. Thus the great tragedies became the tragic victims of the new literary atmosphere of common sense. The conception of poetry as something awful, spiritual, or divine, was almost forgotten during the eighteenth century.

Only a few men, notably Collins, Smart, Cowper, and Blake, some supposedly with traces of insanity in their make-ups, dared to violate the rules and to write in accordance with their own dictates. Those few writers who persisted in writing the poetry of passion had to smuggle their works into the literary world either by disguising them or by pretending to ridicule their efforts.

These changes in the poetic spirit were so drastic that some critics, especially in the nineteenth century, felt that all beauty had been banished. Keats execrated Descartes and Newton for cutting the throat of poetry and Blake damned them. At a dinner party in the year 1817 Wordsworth, Lamb, and Keats, among others, drank a toast which ran: "Newton's

health, and confusion to mathematics." But the critics of the period saw beauty in adherence to the strict rules of versification.

The code for poetry was hopelessly inadequate. The rules of descriptive geometry produce a draftman's sketch, not a work of architecture. As Burns put it, one cannot "hope to reach Parnassus by dint o'Greek." So reason, taking the place of imagination and feeling, produced prosaic and cold poetry. Not until Wordsworth, Coleridge, Keats, and Shelley revolted against the suppression of emotion and the strict rules for verse did true lyric poetry revive.

The poetry that was written during the Newtonian era was filled with appreciations and praise of the works of mathematics and science. The writers found reason, mathematical order and design, and the vast mechanism of nature themes so moving that these replaced the concern for birth, love, and death of insignificant man. No one is so unrestrained in his enthusiasm for the new wonders of the world as is Dryden.

From harmony, from heavenly harmony.
   This universal frame began:
      From harmony to harmony
Through all the compass of the notes it ran,
The diapason closing full in Man.

As from the power of sacred lays
   The spheres began to move,
And sung the great Creator's praise
   To all the blest above;

Famous also are the lines Alexander Pope intended as an epitaph for Newton's tomb in Westminster Abbey:

Nature and Nature's laws lay hid in night;
God said, "Let Newton be", and all was light.

Except in so far as occasional quotations illustrate the new subject matter it is impossible to survey here the great poetry of the age. The extent of the change in the content of English poetry may be appreciated somewhat if one recalls that the major seventeenth century poets of Newton's youth wrote devotional poetry or love lyrics. Almost all of them ignored mathematics and science. The few who happened to touch upon these subjects seemed unaware of the tremendous import of the current developments. For example, Milton, who had paid his respects to Galileo on a trip to the Continent, and lived 32 years after the latter's death, nevertheless retains Ptolemaic theory in his writings. Still others even ridiculed mathematics. In 1663 Samuel Butler wrote in his Hudibras:

In Mathematicks he was greater
Than Tycho Brahe, or Erra Pater:
For he, by Geometrick scale,
Could take the size of Pots of Ale;
Resolve by Signs and Tangents streight,
If Bread or Butter wanted weight;
And wisely tell what hour o' th' day
The Clock doth strike, by Algebra.

After Newton's work ridicule changed to unbounded admiration.

Literature was not the only art to be strongly influenced by the flourishing and almost domineering mathematical spirit of the age. Painting, architecture, landscape gardening, and even furniture design became subject to rigid conventions and explicitly set standards. In painting, the precepts of Sir Joshua Reynolds illustrate the artistic temper of the times. Fidelity to the object painted, subservience of color to idea, and the sacrifice of details to the general and everlasting elements were stressed by him. Moreover, the painter should address himself to the mind and not the eye. In architecture, landscape gardening, furniture and pottery, order, balance, symmetry and strict adherence to well-known simple geometrical forms ruled the day. Art academies formed on the pattern of the successful scientific academies promulgated the criteria of art and exerted great influence in securing adherence and setting the fashion.

Following the changes in the character of literature, painting, and the other arts came the change in the philosophy of aesthetics which rationalized and justified the new attitudes. The new thesis of aesthetics was that art like science is derived from the study and imitation of nature and hence, like nature, was susceptible of mathematical formulation. According to Sir Joshua Reynolds,

> It is the very same taste which relishes a demonstration in geometry that is pleased with the resemblance of a picture to its original and touched with the harmony of music. All these have unalterable and fixed foundations in nature.

Moreover, said Sir Joshua, the essence of beauty is the expression of universal laws.

Just as observation had produced Kepler's laws so the study of nature would reveal the laws of art. Some, however, believed that reason alone, independently of observation, could deduce by the *a priori* geometrical method the mathematical laws of aesthetics, for beauty like truth is apprehended by the rational faculty.

And so men studied nature or applied their rational faculties to reduce art to a system of rules and beauty to a series of characterizing formulas. Precepts for attaining beauty were laid down and analyses were made of the nature of the sublime. It was expected that the search for beauty in Nature would produce not only an abstract ideal of beauty but its chief characteristics. With this knowledge, beautiful works could then be created almost at will, but only by observance of the rules of art so discovered. Unfortunately great art is still not being made on a mass production basis; perhaps this is because no twentieth century industrial tycoon has understood the eighteenth century as yet.

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We have attempted in this article and an earlier one* to illustrate the

*See "The Harmony of the World", Mathematics Magazine, Jan.-Feb., 1954, pp. 127-139. Refer also to the book *Mathematics in Western Culture*, reviewed on page 121 of this issue.

revolution in culture caused by Renaissance and seventeenth century developments involving mathematics. By 1800 the changes were already immeasurable and the impress was just beginning to be exerted. The implications and amplifications of mathematical activities of the sixteenth and seventeenth centuries are still shaping our thoughts as well as our physical mode of living, more and more so as time passes. Indeed the eighteenth century Age of Reason marked merely the introduction of an essentially modern culture as opposed to an earlier ecclesiastical, feudal one.

In general the influence of Kepler, Galileo, Newton and his contemporaries was to initiate one vast intellectual inquiry into the nature of the world, man, society, and almost every institution and custom of man. The age passed on to its successors the rationalistic spirit which it exalted, the ideal of general, all-embracing laws, and the ideal of mathematical certainty. It also launched our civilization on a quest for omniscience, stimulated the desire to organize thought into systems on the mathematical pattern, and imbued a faith in the power of mathematics and science.

Upon the basis of the striking successes which seventeenth century mathematics and science achieved in the fields of astronomy and mechanics the eighteenth century intellectuals asserted the conviction that all of man's problems would soon be solved. Had these men known of the additional marvels science and mathematics were soon to reveal they would have been even more unreserved, were that possible, in their expectations. It is now evident that these thinkers were indulging in unwarranted optimism. However, their conviction was prophetic at least to the extent of a half-truth, for mathematics and science did go on to remake the world if not solve all its problems. Even in those domains where very little progress has been made towards the solution of basic problems the ideals of the Age of Reason still provide the goals and the driving force.

Despite the almost incredible subsequent growth in the influence of mathematical ideas and techniques, popular regard for the subject today is considerably weaker than it was in the eighteenth century. In perusing journals of that age one is struck by the fact that those written for the class in society which now reads such magazines as *Harpers* and the *Atlantic Monthly* contained mathematical articles side by side with literary articles. The educated man and woman of the eighteenth century knew the mathematics of their day, felt obliged to be *au courant* with all important scientific developments, and read articles on them much as the modern man reads articles on politics. These people were as much at home with Newton's mathematics and physics as with Pope's poetry. Even in the seventeenth century we have the instance of Samuel Pepys so much attracted by the scientific activity of this time that at the age of thirty he undertook to learn mathematics. He began, incidentally, with the multiplication table, which he then taught to his wife. In 1681

Pepys was elected president of the Royal Society of London, a post so distinguished that it was also held by Newton for the last 25 years of his life. Today, by contrast, men commonly regarded as educated feel no compulsion to inform themselves even in elementary algebra.

The paradox is perhaps readily resolved. In the eighteenth century education was confined to a small superior group; the few spoke for their age and gave it its intellectual character. The spread of democracy since the French revolution has brought with it the diffusion of knowledge to large masses whose bulk covers the pages of current history. The participation of great masses in cultural activities has temporarily at least concealed the greatly increased application of reason in our present civilization and has obscured the record of genius. For reason is the highest and most difficult form of human activity, and the level of great masses cannot be expected to reach the level of a few geniuses.

It is of course true that the full role of mathematics in present day civilization is less obvious. The deeper and complex applications which are now so common are not readily comprehended even by specialists. Nevertheless the average person with merely the will to apply himself could learn more mathematics and science in an ordinary present day college program than the best of the eighteenth century men could possibly have known. The trouble would seem to be, at least in the United States, that true education for all who can absorb it exists in name only. As a country we pay lip service to education but devote practically no money or energy to the task. Consequently the high schools and colleges educate those who can afford to come and suit the curriculum to the student. To make a bad situation worse, it is the graduates of these schools who become in turn the leaders in education.

These facts account in large part for the tendency of people today to belittle or ignore mathematics, which dominates our age even more than it did the eighteenth century, whereas the learned of that period regarded it as paramount in importance.

New York University

PROBLEMS AND QUESTIONS

Edited by

Robert E. Horton, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new and subject matter questions that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and the size desired for reproduction.

Send all communications for this department to *Robert E. Horton, Los Angeles City College, 855 North Vermont Avenue, Los Angeles 29, California.*

PROPOSALS

215. *Proposed by Norman Anning, Alhambra, California.*

Solve $\operatorname{arc\,cot} x + \operatorname{arc\,cot} y = \operatorname{arc\,cot} z$ where x, y and z are integers such that $x > y > z > 0$.

216. *Proposed by Erich Michalup, Caracas, Venezuela.*

Prove that

$$\sum_{n=1}^{\infty} \frac{16n^2 + 12n - 1}{8(4n+3)(4n+1)(2n+1)(n+1)} = \frac{1}{24}$$

217. *Proposed by Huseyin Demir, Zonguldak, Turkey.*

Prove that a necessary and sufficient condition for the convex polygon $A_1 A_2 A_3 A_4$ to be inscriptable is that:

$$D = \begin{vmatrix} A_1 A_1 & A_1 A_2 & A_1 A_3 & A_1 A_4 \\ A_2 A_1 & A_2 A_2 & A_2 A_3 & A_2 A_4 \\ A_3 A_1 & A_3 A_2 & A_3 A_3 & A_3 A_4 \\ A_4 A_1 & A_4 A_2 & A_4 A_3 & A_4 A_4 \end{vmatrix} = 0$$

where A_{ij} denotes the distance between the vertices A_i and A_j if $j > i$, and $A_j A_i = -A_i A_j$.

218. *Proposed by Ben K. Gold, Los Angeles City College.*

Prove

$$\sum_{i=0}^{K} (-1)^i \binom{K+i}{K} \binom{2K+1}{K-i} = 1$$

219. *Proposed by N. Shklov, University if Saskatchewan.*

Let a straight line through the origin meet the lines $x+y-4=0$ and $x-y-4=0$ in the points A and B respectively. Let M be the midpoint of the segment AB. Determine the locus of M as the line OAB is rotated about O.

220. *Proposed by Thomas F. Mulcrone, St. Charles College, Louisiana.*

Show that for $n \geq 2$ the Farey sequence F_n contains an odd number of terms.

221. *Proposed by E. P. Starke, Rutgers University.*

On a conical surface there is traced a spiral which crosses each of the linear elements at a fixed angle ϕ. Find a simple expression for the length of this spiral between any two of its points.

SOLUTIONS*

Late Solutions

189, 191. *H. M. Feldman, St. Louis, Missouri.*

A Food Index Function

193. [January 1954] *Proposed by Francis J. Weiss, Washington, D. C..*

In comparing the relative food value of certain foodstuffs or the nutritional adequacy of different diets it is customary to express contents of individual nutrients in fractions or percentages of scientifically determined dietary allowances. There are about 40 different nutrients that are considered essential for maintenance of health (carbohydrates, fats, 10 amino acids, 12 vitamins, and the rest minerals and trace minerals) the lack of a single one may eventually be fatal. Consequently, no matter how plentiful the other nutrients are, if a single nutrient for instance lysine or vitamin C or iron is lacking, the diet is inadequate and death may result. Thus the individual nutrients or their fractions are clearly multiplicative which means that its maintenance value is zero, if only a single factor becomes zero.

On the other hand there is no doubt that the individual nutrients contained in a certain quantity of food or in a particular diet are also additive inasmuch as they contribute their share of calories and substances needed for the body. Consequently, if one wishes to compare the food value of certain foods (one quart of milk, one egg, etc.) or of a certain diet (lumberman in Oregon, child in California), one would have to express the combined nutrient ratios in a single expression that would have to be both additive and multiplicative, for instance:

Recommended dietary allowances per day	$A + B + C + D \cdots$	(1)
Actual food intake or food value per gram	$a + b + c + d \cdots$	(2)
Food index figure	$F = \frac{a}{A} \circledcirc \frac{b}{B} \circledcirc \frac{c}{C} \circledcirc \frac{d}{D} \cdots$	(3)

Simultaneous multiplication and addition is indicated by the symbol $\circledcirc$ for lack of a better one.

Is it possible to carry out such combined operations mathematically? Can a Food Index Function

$$f\left[\frac{a}{A}, \frac{b}{B}, \frac{c}{C}, \cdots\right]$$

be developed which possesses the desired multiplicative and additive properties?

Comment by R. B. Herrera, Los Angeles City College.

A function which might meet the requirements of the Food Index function, of being additive and multiplicative, is the following:

$$F\left[\frac{a}{A}, \frac{b}{B}, \cdots, \frac{x}{X}\right] = \frac{k}{(e-1)^n}\left[(e^{a/A}-1)\cdots(e^{x/X}-1)\right]$$

where n is the number of ratios $\frac{a}{A}$.

More concisely, let

$$F = \frac{k}{(e-1)^n}\prod_{i=1}^{n}(e^{a_i/A_i}-1), \quad 0 \le a_i \le A_i, \; A_i \ne 0.$$

Then F has the properties:

1) If a_i is present in required amount, $a_i = A_i$, the factor $\frac{e^{a_i/A_i}-1}{e-1}$, is equal to 1; otherwise the factor is less than 1 and lowers the value of the whole product.

2) If a_i is absent, $a_i = 0$, but $A_i \ne 0$, then $\frac{e^{a_i/A_i}-1}{e-1} = \frac{1-1}{e-1} = 0$, and $F = 0$.

3) For suitable k, a maximum index such as 1000, can be set. Thus if $a_i = A_i$ for all i, the value of F is equal to k, and otherwise lies between 0 and K.

A Dissected Square

194. [March 1954] *Proposed by C. S. Ogilvy, Hamilton College, New York.*

Rearrange the five quadrilaterals and five triangles to form one perfect square:

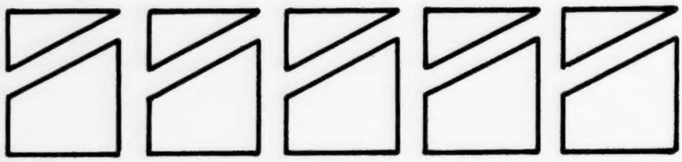


Figure 1

I. *Solution by T. F. Mulcrone, Spring Hill College.* The given quadrilaterals and triangles are obviously sections of squares. If we denote by unity the sides of these squares then the shorter legs of the triangles are ½ unit and their hypotenuses are $\sqrt{5}/2$ units in length.

Hence the one perfect square formed from the sections of the five given squares must have an area of 5 square units, and hence have sides of length $\sqrt{5}$ units, or twice the length of the sections of the original squares. And, if we rotate $\angle DEF$ until FE coincides with AB, then $\angle DEF + \angle ABC = \angle ABC + \angle DEF = 90°$.

Thus, four of the given squares transform into the corner elements of the required square, while the fifth given square is the center portion of the required square, as indicated in the diagram.

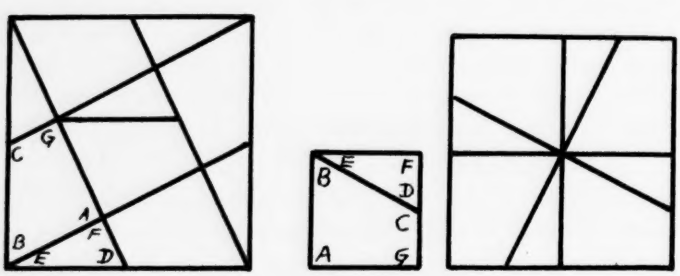


Figure 2

II. *Solution by Henry W. Gould, Portsmouth, Virginia.* Figure on page 107.

Also solved by Leon Bankoff, Los Angeles, California; Richard K. Guy, University of Malaya, Singapore; J. M. Howell, Los Angeles City College (Two Solutions); John Jones Jr., Mississippi Southern College; W. Moser, University of Toronto; W. M. Sanders, Mississippi Southern College; S.

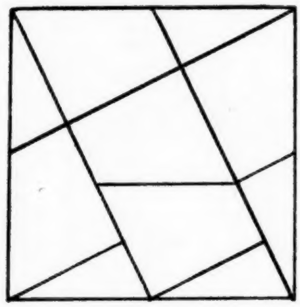


Figure 3

H. Sesskin, Hofstra College, New York; N. Shklov, University of Saskatchewan; C. W. Trigg, Los Angeles City College (Two Solutions); Chih-yi Wang, University of Minnesota; Warren J. Wayne, Butler University; and the proposer.

Leon Bankoff submitted the following references to the problem:

1. Ladies' Diary, Supplement (1803), p. 22.

2. Hutton's Translation of Montucla's edition of Ozanam's Recreations, problem 18, p. 292, Vol. I, (1770).

3. A solution in the Ladies' Diary, 1819, p. 37, Question VI, refers to the figure composed of four congruent right triangles and a square, assembled to form another square, as the "Bride's Chair".

4. Scripta Mathematica, Vol. XIX, No. 4, Dec. 1953, page 270.

5. Sam Loyd's Encyclopedia of 5000 puzzles, Tricks and Conundrums. Lamb Publishing Co., N. Y. 1914, p. 369; p. 227.

6. Henry Ernest Dudeney, Amusements in Mathematics, Thomas Nelson and Sons, Ltd., Toronto and New York, 1917, reprinted in 1951, Problem 147, p. 170.

7. Howard Eves, An Introduction to the History of Mathematics, Rinehart & Co., Inc., New York, 1953, p. 187, 188.

Charles W. Trigg pointed out that the dissection is essentially that shown on page 25 of Sundara Row, Geometric Exercises in Paper Folding, Open Court, (1941).

A Summation

195. [March 1954] *Proposed by Leo Moser, University of Alberta.*

If $[x]$ denotes the greatest integer not exceeding x show that:

$$[1/(\pi^2/6 - \sum_{i=1}^{n} 1/i^2)] = n$$

Solution by Richard K. Guy, University of Malaya, Singapore. It is well known that

$$A = \frac{\pi^2}{6} - \sum_{i=1}^{n} \frac{1}{i^2} = \sum_{i=n+1}^{\infty} \frac{1}{i^2}$$

which lies strictly between

$$\int_n^\infty \frac{dx}{x^2} \quad \text{and} \quad \int_{n+1}^\infty \frac{dx}{x^2}$$

since the function $1/x^2$ decreases steadily. Thus the sum A lies between $1/n$ and $1/(n+1)$. Therefore its reciprocal lies between n and $n+1$ and the integral part of this is n.

Also solved by Leon Bankoff, Los Angeles, California; L. Carlitz, Duke University; M. S. Klamkin, Polytechnic Institute of Brooklyn; W. Moser, University of Toronto; Dennis C. Russell, Birkbeck College, University of London; Chih-yi Wang, University of Minnesota.

Triangle Construction

196. [March 1954] *Proposed by G. W. Courter, Baton Rouge, Louisiana.*

Construct a right triangle given its hypotenuse and a point on it which is the corner of the inscribed square on the hypotenuse.

Solution by C. W. Trigg, Los Angeles City College.

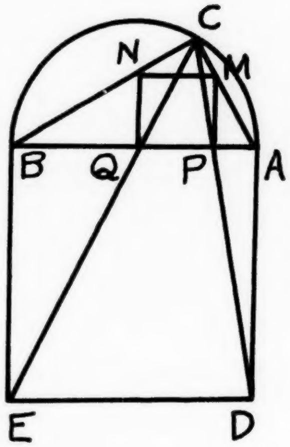
Interpretation **I**. "A side of the square lies on the hypotenuse." Consider a right triangle ABC with side PQ of inscribed square $PQNM$ lying on AB, and with square $ADEB$ external to the triangle. Clearly, C is the homothetic center of the two squares. The construction, given P on AB, is evident, namely: On AB as diameter draw a semicircle. On the other side of AB construct the square $ADEB$. Draw DP extended, meeting the semicircle at C. The required triangle is completed by drawing CA and CB.

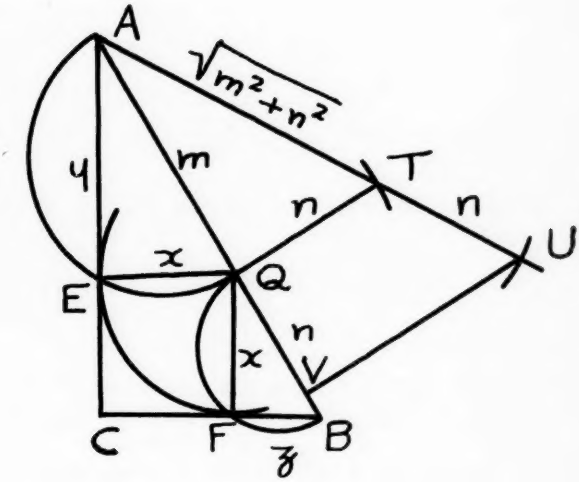
The extension is immediate to the general case: "Construct a triangle given an angle, the opposite side and a point on that side which is a corner of the side of the inscribed square lying on that side."

Interpretation **II**. "Only a corner of the inscribed square lies on the hypotenuse." Consider a right triangle ABC with an inscribed square of side x lying with two sides along the sides of the right angle and a vertex Q dividing the hypotenuse into two parts, m and n. From the figure, $x/m = z/n$ and $x/n = y/m$, so $x^2 = yz = y^2n^2/m^2 = z^2m^2/n^2$. But

$$y^2 = m^2 - x^2 = m^2 - y^2n^2/m^2, \quad \text{so} \quad y^2 = m^4/(m^2+n^2),$$

and $$z^2 = n^2 - x^2 = n^2 - z^2m^2/n^2, \quad \text{so} \quad z^2 = n^4/(m^2+n^2).$$





Therefore,

$$x^2 = m^2n^2/(m^2+n^2) \quad \text{and} \quad x = mn/\sqrt{m^2+n^2}.$$

To construct the triangle, first construct semicircles on AQ and QB as diameters. Construct the fourth proportional ($QV = x$) to $\sqrt{m^2+n^2}$, m and n, as in the figure. With Q as center, and radius x, describe a circle cutting the semicircles in E and F, respectively. Draw AE and BF extended meeting in C.

Also solved by Leon Bankoff, Los Angeles, California; Richard K. Guy, University of Malaya, Singapore; John Jones Jr., Mississippi Southern College; M. A. Kirchberg, Charleston, Illinois; Sam Kravitz, East Cleveland, Ohio; George R. Mott, Rockville Centre, New York; T. F. Mulcrone, St. Charles College, Louisiana; W. M. Sanders, Mississippi Southern College; Hazel Schoonmaker Wilson, Jacksonville State College, Alabama; and the proposer.

A Fibonacci Expression

197. [March 1954] *Proposed by A. S. Gregory, University of Illinois.*

Let the explicit expression for the nth term of a sequence K_n be known. Find an explicit expression for the nth term of a sequence $\{\phi_n\}$ which is defined as follows:

$$\phi_n = \phi_{n-1} + \phi_{n-2} + K_n\,; \quad n = 1,2,3, \cdots$$

with ϕ_0 and ϕ_1 given.

Solution by Murray S. Klamkin, Polytechnic Institute of Brooklyn. From the definition we have:

$$\phi_2 = \phi_1 + \phi_0 + K_2$$

$$\phi_3 = 2\phi_1 + \phi_0 + K_2 + K_3$$

$$\phi_4 = 3\phi_1 + 2\phi_0 + 2K_2 + K_3 + K_4$$

$$\phi_5 = 5\phi_1 + 3\phi_0 + 3K_2 + 2K_3 + K_4 + K_5$$

Let A_r denote the rth term of the Fibonacci sequence 1,1,2,3,5,8, $\cdots$. Explicitly,

$$A_r = \tfrac{1}{2}\left[(1+\sqrt{5})^{r-1} + (1-\sqrt{5})^{r-1}\right]$$

By induction it follows that:

$$\phi_n = A_n\phi_1 + A_{n-1}\phi_0 + A_{n-1}K_2 + A_{n-2}K_3 + \cdots + A_1K_n$$

Also solved by H. W. Gould, Portsmouth, Virginia; Richard K. Guy, University of Malaya, Singapore; W. Moser, University of Toronto; George R. Mott, Rockville Centre, New York; T. F. Mulcrone, St. Charles College, Louisiana; Roy F. Reeves, Columbus, Ohio; Dennis C. Russell, Birkbeck College, University of London; Chih-yi Wang, University of Minnesota; and the proposer.

A Rolling Marble

198. [March 1954] *Proposed by C. W. Trigg, Los Angeles City College.*

A marble of radius r comes down an inclined track and then around a vertical loop of radius R. At what height h above the top of the loop must its center of gravity be at the start in order that it may press against the bottom of the loop with K times the force that it presses against the top? Consider two cases, (a) Rolling without slipping (neglect frictional loss); (b) Sliding without rolling.

Solution by A. L. Epstein, Cambridge Research Center, Boston, Massachusetts. Let the mass of the marble be taken as unity.

a). At height h above the top of the loop the potential energy of the marble is $g(2R+h)$. At the bottom of the loop the marble has potential and kinetic energy of $gr + v_2^2/2 + r^2w_2^2/5$. The last term represents the kinetic energy due to the rotation of the marble, that is $\theta w_2^2/2$ where the moment of inertia of the marble is $2mr^2/5$. But since $rw_2 = v_2$, the energy of the marble is $gr + 7v_2^2/10$. Equating this to the original potential energy we have $v_2^2 = (10g/7)(2R+h-r)$. The force with which the marble presses against the loop here is clearly the gravitational plus the centrifugal force or

$$g + \frac{10g(2R+h-r)}{7(R-r)} \equiv g\left[\frac{27R+10h-17r}{7(R-r)}\right]$$

At the top of the loop the energy of the marble is $g(2R-r)+7v_3^2/10$. Equating this to the original potential energy gives $v_3^2 = 10g(h+r)/7$ Here the marble presses against the loop with a force of

$$\frac{10g(h+r)}{7(R-r)} - g \equiv g\left[\frac{-7R+10h+17r}{7(R-r)}\right].$$

As the force against the bottom of the loop is to be K times the force against the top we have $27R+10h-17r=K(-7R+10h+17r)$ whence

$$h = \frac{R(27-7K)-17r(K+1)}{10(K-1)}.$$

We note that for $0 < h < (7R-17r)/10$ the force at the top of the loop is negative which means that the marble would not reach the top of the loop.

b). In this case the original potential energy is also $g(2R+h)$ but at the bottom of the loop the total energy is $gr+v_2^2/2$ since there is now no rotational energy. This leads to a force against the bottom of the loop of $g[(5R+2h-3r)/(R-r)]$. Likewise at the top of the loop we have the energy $g(2R-r)+v_3^2/2$ yielding a force against the loop of $g[(-R+2h+3r)/(R-r)]$. Putting the force against the loop's bottom equal to K times the force against its top we have $5R+2h-3r = -KR+2Kh+3Kr$ which leads to

$$h= \frac{R(K+5) - 3r(K+1)}{2(K-1)}$$

Here again for $0 < h < (R-3r)/2$ the marble would not reach the top of the loop.

Also solved by the proposer.

Related Trajectories

199. [March 1954] *Proposed by P. D. Thomas, Eglin Air Force Base, Florida.*

Projectiles are fired in a vertical plane at a given initial velocity but varying angles of elevation θ. Of all the pairs of trajectories for θ and $90°-\theta$, where $\theta < 45°$, which give the same range R_θ, show that there is only one pair such that the point P of maximum height attained for θ is the focus of the trajectory at $90°-\theta$. Find the value of θ for which this is true. (Consider trajectories in a vacuum under the influence of gravity).

Solution by M. Morduchow, Polytechnic Institute of Brooklyn. By writing Newton's Second Law for the projectile in the form $\ddot{x}=0$, $\ddot{y}=-g$, integrating under the given initial conditions, and then eliminating

the time, one obtains the well-known parabolic trajectory $y=(\tan\theta)x - (g/2v_0^2)(\sec^2\theta)x^2$, for initial speed v_0 at angle of elevation θ. Setting $y=0$, it is seen that $R_\theta=(v_0^2/g)\sin 2\theta$. Hence, for the same v_0, $R_{90°-\theta}=R_\theta$. The above equation of the path can be written in the equivalent standard form: $(x-R_\theta/2)^2=-(2v_0^2/g)\cos^2\theta\,[y-(v_0^2/2g)\sin^2\theta]$, whence it is seen that the maximum height (attained at $x=R_\theta/2$) will be $y_{max}=(v_0^2/2g)\sin^2\theta$. By substituting $(90°-\theta)$ for θ into the above standard form, it follows that the focus of the parabolic path for angle of elevation $(90°-\theta)$ will be at $x=R_\theta/2$, $y=(v_0^2/2g)\cos^2\theta-(1/4)(2v_0^2/g)\sin^2\theta=(v_0^2/2g)(\cos^2\theta-\sin^2\theta)$. Hence it is required that $(v_0^2/2g)(\cos^2\theta-\sin^2\theta)=(v_0^2/2g)\sin^2\theta$, whence θ must have the (unique) value $\theta=\arctan\sqrt{1/2}\approx 35°16'$.

Also solved by Richard K. Guy, University of Malaya, Singapore; M. S. Klamkin, Polytechnic Institute of Brooklyn; Dennis C. Russell, Birkbeck College, University of London; S. H. Sesskin, Hofstra College, New York; A. Sisk, Maryville College; Chih-yi Wang, University of Minnesota; and the proposer.

A Double Radical Axis

200. [March 1954] *Proposed by Leon Bankoff, Los Angeles, California.*

Chords CD and EF are perpendicular to the diameter AB of a circle. Show that the radical axis of the circle with center at A of radius AC and the circle with center at B of radius BE is a line which is equidistant from CD and EF.

I. *Solution by W. M. Sanders, Mississippi Southern College.* Represent A by $(-a,0)$, B by $(a,0)$, CD by $x=c$, EF by $x=e$ Then the representation of C and E will be $(c,\sqrt{a^2-c^2})$ and $(e,\sqrt{a^2-e^2})$ respectively. The radical axis of the circles $x^2+y^2+2ax=a^2+2ac$ (center at A of radius AC) and $x^2+y^2-2ax=a^2-2ae$ (center at B of radius BE) is the line $4ax=2ac+2ae$. The absolute value of the distance from this line to C is $|(c-e)/2|$ and to E is $|(e-c)/2|$. Hence the desired result is established.

II. *Solution by Richard K. Guy, University of Malaya, Singapore.* Let L, M, O, X be the mid-points of CD, EF, LM and AB respectively. Then the square of the tangent from O to $A(C)$ is

$$\begin{aligned} OA^2 - AC^2 &= (OL + LA)^2 - (AL^2 + LC^2) \\ &= OL^2 + 2OL\cdot LA - (CX^2 - LX^2) \\ &= OL^2 + 2OL\cdot LA - CX^2 + (AX - AL)^2 \\ &= OL^2 + LA(2OL - 2AX + AL) \\ &= OL^2 - LA\cdot BM \end{aligned}$$

The symmetry of this expression in A, B: L, M shows that O is on the radical axis of the circles $A(C)$, $B(E)$, that is, the radical axis lies mid-way between CD and EF.

Also solved by Huseyin Demir, Zonguldak, Turkey; Dennis C. Russell, Birkbeck College, University of London; S. H. Sesskin, Hofstra College, New York; C. W. Trigg, Los Angeles City College; Chih-yi Wang, University of Minnesota, Hazel Schoonmaker Wilson, Jacksonville State College, Alabama; and the proposer.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q 122. The products of the areas of the opposite triangles formed by the sides and diagonals of any quadrilateral are equal. [*Submitted by Vladimir F. Ivanoff.*]

Q 123. If three generators of a right circular cone are

$$\frac{x-1}{1}=\frac{y-2}{2}=\frac{x-3}{-3},\quad \frac{x-1}{-3}=\frac{y-2}{3}=\frac{z+3}{3},\quad \frac{x-1}{11}=\frac{y-2}{8}=\frac{z+3}{9}$$

show that the semi-vertical angle of the cone is $\cos^{-1}\sqrt{3}/3$. [*Submitted by Murray S. Klamkin.*]

Q 124. A convex polygon of twelve sides inscribed in a circle has in some order six sides of length $\sqrt{2}$ and six of length $\sqrt{24}$. What is the radius of the circle? [*Submitted by Norman Anning.*]

Q 125. Show that

$$\frac{(l^2+l+1)(m^2+m+1)(r^2+r+1)(s^2+s+1)(t^2+t+1)}{lmnrst}$$

cannot be less than 729 for all values of l, m, n, r, s and t [*Submitted by Leon Bankoff.*]

Q 126. Show that

$$\sum_{i=0}^{2n} 2^i \binom{4n-i}{2n}\binom{2n+i}{2n}\{4^{n-i}-1\}$$

is identically zero. [*Submitted by "Seriatim".*]

Q 127. Show that the volume of the solid $(x+y)^2+(y+z)^2+(z+x)^2=2$ equals $4\pi\sqrt{2}/3$. [*Submitted by Murray S. Klamkin.*]

ANSWERS

A 122. In the quadrilateral $ABCD$ with diagonals intersecting at P we have $\frac{1}{2}PA \cdot PD \sin APD \cdot \frac{1}{2}PB \cdot PC \sin BPC \equiv \frac{1}{2}PA \cdot PB \sin [\pi - APD] \cdot \frac{1}{2}PD \cdot PC \sin [\pi - BPC]$. Hence the proposition.

A 123. Since the three generators are mutually orthogonal they may be replaced by the x, y and z axes. By symmetry the axis of the cone can be written $x = y = z$. Thus the $\cos\alpha = \sqrt{3}/3$.

A 124. Let AB and BC be a pair of adjacent but unequal sides. Then AC equals r. But $(AB)^2 = 2 + 24 - 2\sqrt{2}\sqrt{24}(-\sqrt{3}/2) = 38$ so $r = \sqrt{38}$.

A 125. The given expression may be written

$$(l + \frac{1}{l} + 1)(m + \frac{1}{m} + 1)(n + \frac{1}{n} + 1)(r + \frac{1}{r} + 1)(s + \frac{1}{s} + 1)(t + \frac{1}{t} + 1)$$

Since the sum of any number and its reciprocal cannot be less than 2, the minimum value of each of the six factors is 3. Hence 3^6 is the minimum value of the expression.

A 126. Generally $\sum_{i=0}^{2n} f(i) = + \sum_{i=0}^{2n} f(2n - i)$ for any f. However, in the given sum we have $\sum_{i=0}^{2n} f(i) = - \sum_{i=0}^{2n} f(2n - i)$. Therefore the sum must be identically zero.

A 127. By rotating the axes, the equation can be transformed into $ax'^2 + by'^2 + cz'^2 = 1$ where a, b and c are the roots of the discriminating cubic

$$\begin{vmatrix} 1-\lambda & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1-\lambda & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1-\lambda \end{vmatrix} = 0$$

$\frac{1}{2}$ is obviously a double root, and by adding the three rows the other root is 2. Thus the volume is $4\pi\sqrt{2}/3$.

QUICKENING THE QUICKIES

Q 105. [March 1954] A quicker way to do this quickie which leads to an immediate generalization is

$$\sum_{1}^{n} r(r+1) = \frac{1}{3}\sum_{1}^{n} [r(r+1)(r+2) - (r-1)(r)(r+1)] = \frac{n(n+1)(n+2)}{3}$$

generalizing we have:

$$\sum_{1}^{n} r(r+1) \cdots (r+k-1) = \frac{1}{k+1}\sum_{1}^{n} [(r)(r+1) \cdots (r+k) - (r-1) \cdots (r+k-1)]$$

$$= \frac{n(n+1) \cdots (n+k)}{k+1}.$$

[*Submitted by M. S. Klamkin*]

CURRENT PAPERS AND BOOKS

Edited by

H. V. Craig

This department will present comments on papers previously published in the MATHEMATICS MAGAZINE, lists of new books, and book reviews.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited.

Communications intended for this department should be sent in duplicate to H. V. Craig, Department of Applied Mathematics, University of Texas, Austin 12, Texas.

*Remarks on the Integration of Products of Functions**

Murray R. Spiegel

FOREWARD: A formula which generalizes the method of integration by parts is given and various applications of it are presented.

. . . . . .

In a recent issue of the Mathematics Magazine [1], R. W. Hakala demonstrated a means by which functions of the form $e^{ax}f(x)$ for suitable $f(x)$ could be integrated. The results given there are special cases of a formula which generalizes the method of integration by parts.

We define $f^{(k)}(x)$ to mean the k'th derivative of $f(x)$ with respect to x; $f^{(0)}(x) \equiv f(x)$, and $g_k(x)$ to mean the k'th integral of $g(x)$ (excluding addition of arbitrary constants); $g_0(x) \equiv g(x)$. Then we have

$$(1) \qquad \int fg\,dx = fg_1 - f^{(1)}g_2 + f^{(2)}g_3 - \cdots (-1)^n f^{(n)} g_{n+1} + (-1)^{n+1} \int f^{(n+1)} g_{n+1}\,dx.$$

The proof of this formula follows immediately upon differentiation of both sides of (1) with respect to x.

In many special cases, the process may be continued indefinitely so that the series

$$\int fg\,dx = fg_1 - f^{(1)}g_2 + f^{(2)}g_3 - \cdots$$

(assumed meaningful) is obtained.

*Essentially a comment on a paper by R. W. Hakala, Math. Mag., Vol. 27, 2, Nov.-Dec., 1953. Ed.

Should $f(x)$ be a polynomial, the series (2) is finite. In the special case where $g(x) = e^{ax}$ and where $f(x)$ is differentiable infinitely often, we have

$$(3) \qquad \int e^{ax} f(x)\,dx = \frac{e^{ax}}{a}\{f - \frac{f^{(1)}}{a} + \frac{f^{(2)}}{a^2} - \cdots\}$$

the case treated by Hakala.

The formulas (1) or (2) provide methods of integration which are useful in that they yield results quickly and easily. However it is of great importance to examine the validity of (2) since the series obtained there may not always be meaningful. We present several illustrations of the use of the general formulas (1) and (2).

Illustration 1

Obtain $\int_a^\infty \frac{\sin x}{x}\,dx$, $a > 0$, in a form suitable for computation.

Solution.

Since $\int_0^\infty \frac{\sin x}{x} = \pi/2$ we may write

$$\int_a^\infty \frac{\sin x}{x}\,dx = \int_0^\infty \frac{\sin x}{x}\,dx - \int_0^a \frac{\sin x}{x}\,dx$$

$$= \frac{\pi}{2} - \int_0^a [1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots]dx$$

$$= \frac{\pi}{2} - a + \frac{a^3a}{3\cdot 3!} - \frac{a^5}{5\cdot 5!} + \cdots.$$

Although this series is convergent for all values of a it is impractical in cases where a is large (for example even when $a \geq 2$). In such cases it is of importance to see whether a more suitable representation may be obtained.

Upon using formula (2) with $f(x) = 1/x$ and $g(x) = \sin x$ one obtains formally

$$\int_a^\infty \frac{\sin x}{x}\,dx = \cos x\,\{-\frac{1}{x} + \frac{2!}{x^3} - \frac{4!}{x^5} + \frac{6!}{x^7} - \cdots\}\Big|_a^\infty$$

$$+ \sin x\,\{-\frac{1}{x^2} + \frac{3!}{x^4} - \frac{5!}{x^6} + \frac{7!}{x^8} - \cdots\}\Big|_a^\infty$$

$$= \frac{\cos a}{a}\,\{1 - \frac{2!}{a^2} + \frac{4!}{a^4} - \frac{6!}{a^6} + \cdots\}$$

$$+ \frac{\sin a}{a}\{\frac{1}{a} - \frac{3!}{a^3} + \frac{5!}{a^5} - \frac{7!}{a^7} + \cdots\}.$$

It is interesting to note in this connection that

$$1 - \frac{2!}{a^2} + \frac{4!}{a^4} - \frac{6!}{a^6} + \cdots$$

is the series obtained by taking the reciprocal of each term in the Maclaurin expansion for $\cos a$ while

$$\frac{1}{a} - \frac{3!}{a^3} + \frac{5!}{a^5} - \frac{7!}{a^7} + \cdots$$

is the series obtained by taking the reciprocal of each term in the Maclaurin expansion for $\sin a$. If we denote these "reciprocal expansions" by $(\cos a)^{[-1]}$ and $(\sin a)^{[-1]}$ respectively we have the result

$$\int_a^\infty \frac{\sin x}{x}\, dx = \frac{\cos a\ (\cos a)^{[-1]} + \sin a\ (\sin a)^{[-1]}}{a}$$

which is easily remembered.

It should be pointed out that the series for $(\cos a)^{[-1]}$ and $(\sin a)^{[-1]}$ do not converge in the usual sense—they are asymptotic series which may be used for "large enough" values of a. For a discussion of asymptotic series [2] is recommended.

Illustration 2

Evaluate $\int e^{ax} \sin bx \ dx$, $a > 0$, $b > 0$.

Solution.

If we let $f(x) = \sin bx$, $g(x) = e^{ax}$ in (2) we obtain

$$\int e^{ax} \sin bx\ dx = e^{ax}\left\{ \frac{\sin bx}{a} - \frac{b \cos bx}{a^2} - \frac{b^2 \sin bx}{a^3} + \frac{b^3 \cos bx}{a^4} + \cdots \right\}$$

$$= e^{ax}\left\{ \frac{a \sin bx - b \cos bx}{a^2} \right\} \left\{ 1 - \frac{b^2}{a^2} + \frac{b^4}{a^4} - \cdots \right\}.$$

$$= \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2}, \qquad a > b.$$

Similarly if we let $f(x) = e^{ax}$, $g(x) = \sin bx$ in (2) we have

$$\int e^{ax} \sin bx\ dx = e^{ax}\left\{ \frac{-\cos bx}{b} + \frac{a \sin bx}{b^2} + \frac{a^2 \cos bx}{b^3} - \frac{a^3 \sin bx}{b^4} + \cdots \right\}$$

$$= e^{ax}\left\{\frac{a \sin bx - b \cos bx}{b^2}\right\}\left\{1 - \frac{a^2}{b^2} + \frac{a^4}{b^4} - \cdots\right\}$$

$$= \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2}, \quad b > a.$$

Thus the same result is reached whether $a > b$ or $b > a$. The derivation by this method fails if $a = b$, unless we allow the use of

$$1 - 1 + 1 - 1 + \cdots = \tfrac{1}{2}$$

which is not valid for usual convergence but is useful in the theory of summable series.

The remark of Hakala that $a > b$ is a "necessary limitation" in order for formula 414 of [3] to hold is not correct. In a similar manner formula 415 does not require $a > b$.

Illustration 3

Evaluate $\int x^3 \sin 2x\, dx$.

Solution.

By letting $f(x) = x^3$, $g(x) = \sin 2x$ in (2) we obtain

$$\int x^3 \sin 2x\, dx$$

$$= (x^3)\left(\frac{-\cos 2x}{2}\right) - (3x^2)\left(\frac{-\sin 2x}{4}\right) + (6x)\left(\frac{\cos 2x}{8}\right) - (6)\left(\frac{\sin 2x}{16}\right)$$

$$= (\sin 2x)\left(\frac{3x^2}{4} - \frac{3}{8}\right) + (\cos 2x)\left(\frac{3x}{4} - \frac{x^3}{2}\right).$$

Illustration 4

Evaluate $\int_0^1 \sqrt{x^2 + 3x + 2}\; dx$

Solution.

Write the integral as

$$\int_0^1 \sqrt{x+1}\,\sqrt{x+2}\; dx$$

By letting $f(x) = \sqrt{x+2}$, $g(x) = \sqrt{x+1}$ in (2) we obtain after some simplification

$$\int \sqrt{x+1}\,\sqrt{x+2}\,dx = (x+2)^{1/2}(x+1)^{3/2} \times$$

$$\left\{ \frac{2}{3} - \frac{2}{1\times3\times5}\left(\frac{x+1}{x+2}\right) - \frac{2}{3\times5\times7}\left(\frac{x+1}{x+2}\right)^2 - \frac{2}{5\times7\times9}\left(\frac{x+1}{x+2}\right)^3 - \cdots \right\}$$

Hence

$$\int_0^1 \sqrt{x^2+3x+2}\,dx$$

$$= \sqrt{24}\left\{ \frac{2}{3} - \frac{2}{1\times3\times5}\left(\frac{2}{3}\right) - \frac{2}{3\times5\times7}\left(\frac{2}{3}\right)^2 - \frac{2}{5\times7\times9}\left(\frac{2}{3}\right)^3 - \cdots \right\}$$

$$- \sqrt{2}\left\{ \frac{2}{3} - \frac{2}{1\times3\times5}\left(\frac{1}{2}\right) - \frac{2}{3\times5\times7}\left(\frac{1}{2}\right)^2 - \frac{2}{5\times7\times9}\left(\frac{1}{2}\right)^3 - \cdots \right\}$$

involving series which converge rapidly.

. . . . .

In conclusion it may be said that many other applications of the general formula (2) are available and will be left to the imagination of the reader.

References

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Rensselaer Polytechnic Institute

An introduction to Plane Projective Geometry. By E. J. Hopkins and J. S. Hails, Oxford University Press, London, pp. 6-276. $5.50.

This elementary text in plane projective geometry has been written for the benefit of the advanced high school senior or college freshman who desires to amplify his knowledge of mathematics. As stated by the authors in the preface, only a rudimentary knowledge of elementary analytic geometry exclusive of the theory of conic sections, is presupposed. A logical development of the subject matter is not given, as it was felt by the authors that the emphasis of mathematical ideas and concepts would be more beneficial for such a student. The book is a mixture of

synthetic and analytic plane projective geometry. In the development of the subject matter, very little use is made of metric concepts. To give the student a feeling for the subject, various projective configurations are intuitively introduced by means of Euclidean geometry. Thereafter these ideas are rigorously defined by means of projective coordinates. The authors have selected for problems various examination questions given in tests by some of the leading colleges and universities in Great Britain. On page 273, is a list of answers to some of the problems in the text, and there is a short index to the subject matter of the book on pages 274-6.

The text consists of sixteen chapters. In the first two chapters are discussed the fundamental elements of projective geometry together with some of the axioms of incidence, and then orthogonal projection is studied for the purpose of preparing the student for the definition of general projection. In Chapter III, perspectivities and general projections together with some of their properties are considered. Chapter IV contains the definition of projective coordinates and the analytic proof of Desargues' Theorem and its converse. The next two chapters are concerned with the study of projectivities in one and two dimensions. The Fundamental Theorem of Projective Geometry and its consequences are proved by means of the properties of real numbers. The analytic development of cross ratio is given in Chapter VII. In Chapters VIII, IX, X, XII, XIII, the projective theory of conic sectjons is developed including the classical theorems of Steiner, Pascal, Brianchon, and Pappus. In Chapter XI, complete quadrangles and quadrilaterals are studied including the theory of harmonic ranges and Desargues' Theorem for conic sections. Chapters XIV and XV are concerned with the development of Euclidean geometry both in the real and complex domains. Topics include the circular points at infinity, isotropic lines, Laguerre's definition of angle, and the fundamental metric properties of conic sections. The book closes with Chapter XVI in which is developed the theory of polar reciprocation and its applications to the inversive geometry of Moebius.

On the whole, the book is well written and the material is carefully presented. In the opinion of the reviewer, the authors have fulfilled their aims. This is a nice elementary introduction to the subject of synthetic and analytic projective geometry. Possibly a student could use such a book as a preview to such classics on the subject as Projective Geometry by Veblen and Young, and Principles of Geometry by H. F. Baker. However in the final analysis, the reviewer feels that the student who desires to master any particular field of mathematics, should endeavor to study the real classics in that particular field.

John De Cicco

Mathematics in Western Culture. By Morris Kline, Oxford University Press, 1953, 472 pp., $7.50.

The object of this book is to support the thesis that mathematics has been a major cultural force in Western civilization. Accordingly the book shows how various developments in mathematics proper in turn influenced developments in logic, astronomy, philosophy, painting, music, religious thought, literature, and the social sciences. The conversion of mathematics by Greek philosophers into an abstract, deductive system of thought, the Greek and modern doctrine that nature is mathematically designed, the use of mathematics by Hipparchus and Ptolemy and later by Copernicus and Kepler to erect the most impressive astronomical theories, the development of a mathematical system of perspective by Renaissance painters who sought to achieve realism, the deduction by Galileo, Newton, and others of universal scientific laws which "united heaven and Earth", the reorganization of philosophy, religion, literature, and the social sciences in the Age of Reason, the rise of a statistical view of natural laws consequent upon the success of statistical procedures in the physical and social sciences, the effect of the creation of non-Euclidean geometry upon the belief in truth and on the common understanding of the nature of mathematics, and mathematics as an art are some of the illustrations of the cultural influences of mathematics.

These topics are discussed in connection with one or another of the major mathematical creations which have been introduced since ancient times and hence accompany some presentation of the concepts of Euclidean geometry, trigonometry, projective geometry, coordinate geometry, statistics, the theory of probability, transfinite numbers, non-Euclidean geometry, and other mathematical subjects. The treatment of the mathematics proper emphasizes the ideas themselves. Very little attention is given to the techniques, which are of value only to those who intend to use mathematics in later life.

Though the book is by no means a history of mathematics, the sequence of topics is the historical one. This order happens to be the most convenient for the logical presentation of the subject and is the natural way of examining how the mathematical ideas arose, what the motivations were for investigating these ideas, and how the mathematical creations in turn altered the course of other branches of our culture. An important by-product is that the reader may get some indication of how mathematics has developed, how its periods of activity and quiescence have been related to the history of Western civilization, and how mathematics has been shaped by the civilizations which preceded the modern Western period.

An attempt has been made to make the presentation lively and readable and thereby attract students and laymen to mathematics.

On the whole the book tries to answer the question: What contributions has mathematics made to Western life and thought aside from the techniques which serve the engineer? The author believes that the answer given to this question should constitute the course in mathematics for liberal arts students who do not intend to use techniques in professional work. The book might serve as the basis for such a course.

Morris Kline

Theory of Functions of a Complex Variable, Vol. I. By Constantin Carathéodory, New York, Chelsea Publishing Company, 1954, 314 pp. $4.95.

The book begins with a treatment of Inversion Geometry (geometry of circles). This subject, of such great importance for Function Theory, is taught in great deatil in France, whereas in German-language and English-language universities it is usually dealt with in much too cursory a fashion. It seems to me, however, that this branch of geometry forms the best avenue of approach to the Theory of Functions; it was, after all, his knowledge of Inversion Geometry that enabled H. A. Schwarz to achieve all of his celebrated successes.

Lindeloef's Principle, normal families, Ostrowski's idea of using spherical distance - I have not hesitated to exploit systematically all of the advantages these various methods have to offer.

C. Caratheodory

(Note: Inasmuch as Professor Carathéodory died before his book appeared, the above review, offered for publication in the MATHEMATICS MAGAZINE, is taken from Carathéodory's own preface to his book.)

Chelsea Publishing Company

Theory of Games and Statistical Decisions. By David Blackwell and M. A. Girshick, John Wiley & Sons, New York, 355 pp., $7.50.

By restricting their subject matter to discrete probability distribution, the authors have avoided measure-theoretic difficulties without sacrificing any of the basic ideas that underlie the modern theory of statistical decisions. A great deal of new material is included which they found necessary to develop in order to present a unified treatment of the field.

In offering new techniques in statistical analysis, and providing a fresh approach to the problems of design and analysis of experiments, Blackwell and Girshick deal with such topics as games in normal form, values and optimum strategies in games, general structure of statistical games, utility and principles of choice, classes of optimal strategies, and fixed sample size games with finite Omega and A. Other chapter headings cover the invariance principle in statistical games, sequential games, Bayes and minimax sequential procedures when both Omega and *A* are finite, estimation, and comparison of experiments.

Richard Cook

Psychology of Invention in the Mathematical Field. By Jacques Hadamard, Dover Publications, 920 Broadway, New York 10, N.Y., $1.25.

Most of the great scientific innovators - men like Einstein, Galton, Euler, Descartes, and Hilbert, for example - have first "seen" their ideas as images, rather than as words or mathematical symbols, according to a survey conducted by the noted mathematician, Jacques Hadamard in his book.

The 158-page book investigates the role of the unconscious, the relation between intuition and logic, and other aspects of scientific invention, and quotes extensively from the letters and writings of more than 40 outstanding men in all fields. A few samples:

Einstein: "The psychical entities which seem to serve as elements in (my) thought are certain signs and more or less clear images which can be 'voluntarily' reproduced and combined. The above mentioned elements are, in my case, of visual and some of muscular type."

Galton, the English geneticist: "I do not so easily think in words as otherwise. It often happens that after being hard at work, and having arrived at results that are perfectly clear and satisfactory to myself, when I try to express them in language I feel that I must begin by putting myself on quite another intellectual plane. That is one of the small annoyances of my life."

Titchener, the English psychologist: "Reading any work, I arrange the facts or arguments in some visual pattern, fearing that, as one gets older, one tends also to become more and more verbal in type."

On the other hand, G. Polya, the distinguished mathematician and teacher, writes: "I believe that the decisive idea which brings the solution of a problem is rather often connected with a well-turned word or sentence. The right word, the subtly appropriate word, helps us to recall the mathematical idea (and) gives things a physiognomy."

"The Psychology of Invention in the Mathematical Field" has been republished as an aid to scientists, mathematicians, and engineers engaged in creative research.

Dover Publications

The Foundations of Statistics. By Leonard J. Savage, John Wiley & Sons, Inc., 440 Fourth Avenue, New York 16, New York, 1954, 294 pp. $6.00.

Concentrating on the human aspects of probability as it reflects itself in economic behavior, Dr. Savage reopens the question of the personalistic view and clarifies the pros and cons surrounding it. His early chapters are concerned with the foundations at a relatively deep level. In these, the author develops, explains, and defends an abstract theory of the behavior of a highly idealized person faced with uncertainty. This theory is shown to have as its implications a theory of personal probability, corresponding to the personalistic view of

probability basic to the book, as well as a theory of utility.

In the second part of the book, Dr. Savage moves to a shallower level of the foundations of statistics in what he calls a transition from "pre-statistics to statistics proper." He recognizes that the theory previously developed is too highly idealized for immediate application. Appropriate compromises are therefore explored through an analysis of the inventions and ideas of the British-American school. By this means, the author hopes to show that mutual support and clarification exist between seemingly incompatible systems.

Dr. Savage, assistant professor of statistics at the University of Chicago, is also a consultant for researchers in a variety of fields. His work is already known to those working in statistics, economics, psychology, philosophy, and mathematics.

Richard Cook

Analytic Geometry, second edition. By Edward S. Smith, Meyer Salkover, and Howard K. Justice, March 1953, John Wiley & Sons, N. Y., 306 pp., $4.00.

Containing the material covered in a standard but comprehensive course in plane and solid analytic geometry, the book provides a solid background for further study in applied and theoretical mathematics. The authors have improved the treatment of the angle between two lines, excluded values, asymptotes, bisector of an angle, tangent to a circle, radical axes, and parametric form of the equations of a line. Other changes include: revised problems, new articles on analytical proofs of geometrical theorems and on cylindrical and spherical coordinates, and a four-place table of trigonometric functions, with angles in radians and degrees.

Richard Cook

OUR CONTRIBUTORS

(Continued from back of contents.) graduate of the University of Chicago (A.B. '29; M.S. '31), he remained there until 1933 as a special assistant to the late Professor Herbert E. Slaught. His main interests are Number Theory and the improvement of teaching of Mathematics in the secondary schools and colleges.

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